Identification and Linear Estimation of General Dynamic Programming Discrete Choice Models

Cheng Chou*

Department of Economics, University of Leicester

October 14, 2016

Abstract

This paper studies the nonparametric identification and estimation of the structural parameters, including the per period utility functions, discount factors, and state transition laws, of general dynamic programming discrete choice (DPDC) models. I show an equivalence between the identification of general DPDC model and the identification of a linear GMM system. Using such an equivalence, I simplify both the identification analysis and the estimation practice of DPDC model. First, I prove a series of identification results for the DPDC model by using rank conditions. Previous identification results in the literature are based on normalizing the per period utility functions of one alternative. Such normalization could severely bias the estimates of counterfactual policy effects. I show that the structural parameters can be nonparametrically identified without the normalization. Second, I propose a closed form nonparametric estimator for the per period utility functions, the computation of which involves only least squares estimation. Neither the identification nor the estimation requires terminal conditions, the DPDC model to be stationary, or having a sample that covers the entire decision period. The identification results also hold for the DPDC models with unobservable fixed effects.

*I am thankful to my advisor Geert Ridder for his valuable guidance and advice throughout my graduate school years. I also benefitted from discussions with Nikhil Agarwal, Dalia Ghanem, Gautam Gowrisankaran, Han Hong, Cheng Hsiao, Yu-Wei Hsieh, Yingyao Hu, Roger Moon, Whitney Newey, Matt Shum, Botao Yang, and participants in the econometrics seminar at USC, UCSD and UC Davis. Early versions of the paper were presented at the 2015 World Congress of the Econometric Society, the 2015 California Econometrics Conference, and the 2016 Bristol Econometrics Workshop.

Keywords: Dynamic discrete choice; linear GMM; identification; fixed effects

1 Introduction

The existing identification results (Magnac and Thesmar, 2002; Blevins, 2014) and estimation methods for (non)stationary DPDC models are both conceptually complicated and numerically difficult due to the complexity of (non)stationary dynamic programming that is a recursive solution method. This paper will show that the identification of (non)stationary DPDC models and their estimation can be greatly simplified, because we will show that the identification of general DPDC models is equivalent to the identification of a linear GMM system. So the identification of DPDC models can be understood from the familiar rank conditions in linear models. Moreover, the per period utility functions and discount factors can be estimated by a closed form linear estimator, which does not involve numerical optimization. Monte Carlo studies show that my estimator is numerically stabler and substantially faster than the existing estimators, including nested fixed point (NFXP) algorithm, pseudo-maximum likelihood (PML), and nested pseudo-likelihood (NPL) algorithm. Besides its numerical advantage, my estimator has wider application, because it does requires terminal conditions, the DPDC model to be stationary, or having a sample that covers the entire decision period. The equivalence between the identification of DPDC and linear GMM also holds for the DPDC models with unobservable fixed effects that are correlated with the observable state variables.

The idea of linear identification and estimation is inspired by the econometric literature on dynamic game models. Pesendorfer and Schmidt-Dengler (2008), Bajari, Chernozhukov, Hong, and Nekipelov (2009), Bajari, Hong, and Nekipelov (2010) show that the Markovian equilibria of dynamic games with discrete choices can be equivalently written as a system of equations linear in the per period utility functions. Hence the identification of per period utility functions in dynamic game models is similar to the identification of a linear GMM system. Moreover, the per period utility functions can then be estimated by least squares. As a special case of the dynamic game with discrete choices, the identification and estimation of infinite horizon stationary single agent DPDC models can also be addressed using the equivalence to a linear GMM system (Pesendorfer and Schmidt-Dengler, 2008; Srisuma and Linton, 2012). Because the equivalence to a linear GMM has greatly simplified our understanding of the identification of stationary DPDC models and their estimation, a natural question is if such an equivalence exists for general DPDC models, especially finite horizon nonstationary DPDC models. Finite horizon models are common in labor economics, since households live for a finite time. This paper addresses this question.

The DPDC model studied in this paper is general in three ways. First, the decision horizon can be finite or infinite. Second, all structural parameters, including per period utility functions, discount factors and transition laws, are allowed to be time varying. Third, we allow for unobservable fixed effects in the DPDC models. The fixed effects could be correlated with the observable state variables. Fourth, we do not assume that the per period utility function associated with one particular alternative is known, or is normalized to be a known constant. This feature is important, because normalization of the per period utility function will bias counterfactual policy predictions. In my empirical example, we consider the married woman's counterfactual labor force participation probability if her husband's income growth became slower. According to the Current Employment Statistics, the average hourly real earnings in 2008 (January) and 2016 (June) grow at 3.7% and 1.7% in the United States. According to the Current Population Survey, the labor force participation rates in 2008 (January) and 2016 (January) are 66.2% and 62.7% in the United States. So it would be interesting to see how does female labor force participation rates change as husbands' earnings growth becomes slower. Without normalization assumption, we found that the counterfactual labor force participation probabilities would be lower than the actual ones for those women whose working experience is below the median. However, with normalization assumption, the counterfactual women's labor force participation probabilities would be close to their actual ones, suggesting no effects on female labor force participation from the slower earning growth.

The normalization derives from the analogy between dynamic and static choice. In static discrete choice the conditional choice probabilities (CCP) only depend on the differences between the payoffs of alternatives. So one can change payoffs of alternatives so long as their differences are not changed. This ambiguity motivates the normalization of the payoff of one alternative (Magnac and Thesmar, 2002; Bajari, Benkard, and Levin, 2007; Pesendorfer and Schmidt-Dengler, 2008; Bajari, Chernozhukov, Hong, and Nekipelov, 2009; Blevins, 2014). However, normalization in dynamic discrete choice models is not innocuous for counterfactual policy predictions.

This point has been mentioned recently by some authors in a variety of settings, e.g. Norets and Tang (2014); Arcidiacono and Miller (2015); Aguirregabiria and Suzuki (2014); Kalouptsidi, Scott, and Souza-Rodrigues (2015).[1] The intuition is that in a dynamic discrete choice model, a forward-looking individual's current choice depends on future utility. This future utility depends on the per period utility functions of all alternatives. Consider the normalization of setting the per period utility of the first alternative to be zero for all states. Such a normalization will distort the effects of the current choice on future utility, because the per period utility of the first alternative does not depend on the state. When one consider counterfactual interventions, the effects of the current choice on counterfactual future payoff will be also distorted, hence the counterfactual choice probability will be biased.

Without imposing a normalization, we provide two alternative ways to identify the per period utility functions and discount factors. One is to assume that there are excluded state variables that do not affect per period utilities but affect state transitions. When excluded state variables are not available, another way is to assume that per period utility function is time invariant but that state transition laws are time varying. The excluded variables restriction has been used to identify discount factors in exponential discounting (Ching and Osborne, 2015) and hyperbolic discounting (Fang and Wang, 2015), but it has not been used to identify per period utility functions in general DPDC models. The closest work is Aguirregabiria and Suzuki's (2014) study of market entry and exit decisions, where the per period utility function is equal to the observable revenue net of unobservable cost. Assuming that the firms' dynamic programming problem is stationary, and the discount factor is known, they use exclusion restrictions to identify the cost function. However they do not consider the identification of the discount factor and of nonstationary DPDC models. Let us consider a binary choice model to explain the intuition why the exclusion restrictions can identify the per period utility function without normalization. The observable CCP is determined by the difference between the payoffs of the two alternatives. In DPDC model, such a payoff difference is the sum of the difference between per period utility functions and the difference between the discounted continuation value functions. Exclusion restrictions create "exogenous" variation that can identify the value functions from the CCP. The identification of the per period

[1] We provide two propositions in the Supplemental Material showing the misleading consequence of normalization for counterfactual analysis.

utility functions follows from the Bellman equation.

When there is unobservable fixed effect that is correlated with the observable state variables in the dynamic discrete choice model, we show how to identify the model with control variables approach. Using control variables, we show that the identification issue is still equivalent to the identification in linear GMM models. Unlike Kasahara and Shimotsu (2009), the unobservable heterogeneity (here, fixed effect) is not discrete type. More importantly, we are interested in the identification of structural parameters, such as per period utility functions and discount factors, rather the type-specific CCP in Kasahara and Shimotsu (2009). Hu and Shum (2012) studies the identification of CCP and state transition law in the presence of continuous unobservable heterogeneity, but they do not consider the identification of structural parameters.

Using the equivalence to linear GMM, the estimation of DPDC models becomes so simple that the per period utility functions and discount factors can be estimated by a closed form linear estimator after estimating the conditional choice probabilities (CCP) and the state transition distributions. The implementation of our linear estimator is simple because only basic matrix operations are involved. Our linear estimator can be applied to situations where the agent's dynamic programming problem is nonstationary, the panel data do not cover the whole decision period, and there are no terminal conditions available. Such simplicity in computation and flexibility in modeling are desirable in practice, because the existing estimation algorithms (Rust, 1987; Hotz and Miller, 1993; Aguirregabiria and Mira, 2002; Su and Judd, 2012) depend on complicated numerical optimization and/or iterative updating algorithms, and many of them cannot be applied when the dynamic programming problem is nonstationary and no terminal conditions are available.

In section 2, we develop the DPDC model of which per period utility functions, state transition distributions and discount factors are allowed to be time varying. In section 3, we show the identification and estimation of a four-period DPDC model. We then show the identification of general DPDC models in section 4. In section 5, we show that the DPDC model can be estimated by simple closed-form estimators, which do not involve numerical optimization. Numerical experiments in section 6 are conducted to check the performance of our estimators.[2] As an empirical exam-

[2]Computation for the work described in this paper was supported by the University of Southern California's Center for High-Performance Computing (http://hpcc.usc.edu).

ple, we estimate a female labor force participation model in section 7 and estimate the woman's counterfactual labor force participation probability when her husband's income growth becomes lower. The last section concludes the paper.

Notation. Let X, Y and Z be three random variables. We write $X \perp\!\!\!\perp Y$ to denote that X and Y are independent, and write $X \perp\!\!\!\perp Y|Z$ to denote that conditional on Z, X and Y are independent. Let $f(X)$ be a real function of X. If X can take values $x_1, \ldots, x_{d_x}$, we use f to denote the d_x-dimensional vector $(f(x_1), \ldots, f(x_{d_x}))^\intercal$. For a real number a, let $a_n \equiv (a, \ldots, a)^\intercal$ be an n-dimensional vector with entries all equal to a.

2 Dynamic programming discrete choice model

2.1 The model

We restrict our attention to the binary choice case. The extension to multinomial choice is in Remark 4 in section 4. In each period t, an agent makes a choice $D_t \in \{0, 1\}$ based on a vector of state variables $\Omega_t \equiv (S_t, \varepsilon_t^0, \varepsilon_t^1)$. Researchers only observe the choice D_t and the state variable S_t. The choice D_t affects both the agent's instantaneous utility in period t and the distribution of the next period state variable Ω_{t+1}. Assumption 1 restricts the instantaneous utility to be additive in the unobserved state variables $\varepsilon_t \equiv (\varepsilon_t^0, \varepsilon_t^1)^\intercal$. Assumption 2 assumes that the state variable Ω_t is a controlled first-order Markov process. Both assumptions are standard in the literature.

Assumption 1. *The agent receives instantaneous utility $u_t(\Omega_t, D_t)$ in period t, and*

$$u_t(\Omega_t, D_t) = D_t \cdot (\mu_t^1(S_t) + \varepsilon_t^1) + (1 - D_t) \cdot (\mu_t^0(S_t) + \varepsilon_t^0).$$

We call $\mu_t^{\mathsf{d}}(S_t)$ the (structural) per period utility function in period t.

Assumption 2. *For any $s < t$, $\Omega_{t+1} \perp\!\!\!\perp (\Omega_s, D_s)|(\Omega_t, D_t)$.*

Let $T_* \leq \infty$ be the last decision period. In each period t, the agent makes a sequence of choices $\{D_t, \ldots, D_{T_*}\}$ to maximize the expected remaining lifetime utility,

$$u_t(\Omega_t, D_t) + \sum_{r=t+1}^{T_*} \left(\prod_{j=t}^{r-1} \delta_j \right) \mathrm{E}_{\Omega_r}[u_r(\Omega_r, D_r)|\Omega_t, D_t],$$

where $\delta_t \in [0, 1)$ is the discount factor in period t. The agent's problem is a Markov decision process, which can be solved by dynamic programming. Let $V_t(\Omega_t)$ be the value function in period t. The optimal choice D_t solves the Bellman equation,

$$V_t(\Omega_t) = \max_{\mathtt{d}\in\{\mathtt{0},\mathtt{1}\}} \mu_t^{\mathtt{d}}(S_t) + \varepsilon_t^{\mathtt{d}} + \delta_t \operatorname{E}_{\Omega_{t+1}}[V_{t+1}(\Omega_{t+1})|S_t, \varepsilon_t, D_t = \mathtt{d}]. \tag{2.1}$$

Without further restriction about the state transition distribution, the continuation value $\operatorname{E}_{\Omega_{t+1}}[V_{t+1}(\Omega_{t+1})|S_t, \varepsilon_t, D_t]$ is non-separable from the unobserved state variable ε_t. To avoid dealing with non-separable models, we make the following assumption.

Assumption 3. (i) *The sequence $\{\varepsilon_t\}$ is independent and identically distributed.*

(ii) *For each period t, $S_t \perp\!\!\!\perp (\varepsilon_t, \varepsilon_{t+1})$.*

(iii) *For each period t, $S_{t+1} \perp\!\!\!\perp \varepsilon_t|(S_t, D_t)$.*

The assumption is standard in the literature, but we want to emphasize the implied limitations. Assumption 3.(i) implies that the unobserved state variable ε_t does not include the unobserved heterogeneity that is constant or serially correlated over time. For example, suppose $\varepsilon_t^{\mathtt{d}} = \eta + \omega_t^{\mathtt{d}}$, where η is unobserved heterogeneity, and $\omega_t^{\mathtt{d}}$ is serially independent utility shock. Then $\varepsilon_t^{\mathtt{d}}$ becomes serially correlated. Moreover, if η is fixed effect that is correlated with the observed state variable S_t, Assumption 3.(ii) is violated. If conditional on (S_t, D_t), the unobserved heterogeneity η can still affect the distribution of S_{t+1}, Assumption 3.(iii) is violated.

We will return to the unobservable heterogeneity issue in subsection 4.3. There, we consider the extension of the model allowing for fixed effect. It turns out that our identification and estimation results can still be applied even in this general error specification.

Applying Assumption 3, it can be verified that

$$\operatorname{E}_{\Omega_{t+1}}[V_{t+1}(\Omega_{t+1})|S_t, \varepsilon_t, D_t] = \operatorname{E}_{S_{t+1}}[v_{t+1}(S_{t+1})|S_t, D_t],$$

where

$$v_{t+1}(S_{t+1}) \equiv \operatorname{E}_{\varepsilon_{t+1}}[V_{t+1}(S_{t+1}, \varepsilon_{t+1})|S_{t+1}] \tag{2.2}$$

is called the *ex ante value function* in the literature. Because the conditional expectations $\operatorname{E}_{S_{t+1}}(\cdot|S_t, D_t = \mathtt{0})$ and $\operatorname{E}_{S_{t+1}}(\cdot|S_t, D_t = \mathtt{1})$ as well as their difference will be frequently used, define

$$\begin{aligned} \operatorname{E}_{t+1}^{\mathtt{d}}(\cdot|S_t) &\equiv \operatorname{E}_{S_{t+1}}(\cdot|S_t, D_t = \mathtt{d}),\ \mathtt{d} \in \{\mathtt{0},\mathtt{1}\}, \\ \operatorname{E}_{t+1}^{\mathtt{1/0}}(\cdot|S_t) &\equiv \operatorname{E}_{S_{t+1}}(\cdot|S_t, D_t = \mathtt{1}) - \operatorname{E}_{S_{t+1}}(\cdot|S_t, D_t = \mathtt{0}). \end{aligned} \tag{2.3}$$

Define the *alternative specific value function* (ASVF) $v_t^{\mathtt{d}}(S_t)$,

$$\begin{aligned} v_t^{\mathtt{d}}(S_t) &\equiv \mu_t^{\mathtt{d}}(S_t) + \delta_t \operatorname{E}_{\Omega_{t+1}}[V_{t+1}(\Omega_{t+1})|S_t, \varepsilon_t^{\mathtt{0}}, \varepsilon_t^{\mathtt{1}}, D_t = \mathtt{d}] \\ &= \mu_t^{\mathtt{d}}(S_t) + \delta_t \operatorname{E}_{t+1}^{\mathtt{d}}[v_{t+1}(S_{t+1})|S_t]. \end{aligned} \tag{2.4}$$

Using the notation of the ASVF, the Bellman equation (2.1) becomes

$$V_t(S_t, \varepsilon_t) = \max_{\mathtt{d} \in \{\mathtt{0},\mathtt{1}\}} v_t^{\mathtt{d}}(S_t) + \varepsilon_t^{\mathtt{d}}, \tag{2.5}$$

and the agent's decision rule is simply

$$D_t = 1(\varepsilon_t^{\mathtt{0}} - \varepsilon_t^{\mathtt{1}} < v_t^{\mathtt{1}}(S_t) - v_t^{\mathtt{0}}(S_t)). \tag{2.6}$$

Let $G(\cdot)$ be the CDF of $\tilde{\varepsilon}_t = \varepsilon_t^{\mathtt{0}} - \varepsilon_t^{\mathtt{1}}$. In terms of G, the CCP $p_t(S_t) = \operatorname{P}(D_t = \mathtt{1}|S_t)$ is

$$\begin{aligned} p_t(S_t) &= G(v_t^{\mathtt{1}}(S_t) - v_t^{\mathtt{0}}(S_t)) \\ &= G(\mu_t^{\mathtt{1}}(S_t) - \mu_t^{\mathtt{0}}(S_t) + \delta_t \operatorname{E}_{t+1}^{\mathtt{1}/\mathtt{0}}[v_{t+1}(S_{t+1})|S_t]). \end{aligned} \tag{2.7}$$

When the CDF G is unknown, even the ASVF difference $v_t^{\mathtt{1}}(S_t) - v_t^{\mathtt{0}}(S_t)$ cannot be identified, let alone the per period utility functions $\mu_t^{\mathtt{0}}$ and $\mu_t^{\mathtt{1}}$. Suppose that the CDF G is known, the absolute level of $\mu_t^{\mathtt{0}}(S_t)$ and $\mu_t^{\mathtt{1}}(S_t)$ cannot be identified. Take $\delta_t = 0$ for example, for any constant $c \in \mathbb{R}$,

$$p_t(S_t) = G(\mu_t^{\mathtt{1}}(S_t) - \mu_t^{\mathtt{0}}(S_t)) = G([\mu_t^{\mathtt{1}}(S_t) + c] - [\mu_t^{\mathtt{0}}(S_t) + c]).$$

The following assumption is to address these concerns.

Assumption 4. (i) *The CDF $G(\cdot)$ of $\tilde{\varepsilon}_t \equiv \varepsilon_t^{\mathtt{0}} - \varepsilon_t^{\mathtt{1}}$ and $\operatorname{E}(\varepsilon_t^{\mathtt{0}})$ are known. Moreover, $\tilde{\varepsilon}_t$ is a continuous random variable with real line support, and the CDF G is strictly increasing.*

(ii) *The observable state variable S_t is discrete with time invariant support $\mathcal{S} = \{s_1, \ldots, s_{d_s}\}$.*

(iii) *(Normalization). For every period t, let $\mu_t^{\mathtt{0}}(s_1) = 0$.*

Note that besides the presence of the unknown ex ante value function $v_{t+1}(S_{t+1})$, the CCP formula (2.7) is similar to the CCP in the binary static discrete choice model studied by Matzkin (1992), in which the CDF G can be nonparametrically identified. With the "special regressor" and the median assumption as assumed in Matzkin (1992), the CDF G of $\tilde{\varepsilon}_t$ can be identified by following Matzkin's arguments (see also page 205 of Aguirregabiria, 2010).

The normalization in Assumption 4.(iii) differs from the commonly used normalization by letting

$$\mu_t^{0}(s_1) = \mu_t^{0}(s_2) = \cdots = \mu_t^{0}(s_{d_s}) = 0, \quad \forall t. \tag{2.8}$$

The normalization (2.8) implies that the per period utility of alternative $\mathtt{0}$ does not vary with respect to the values of the state variable S_t. It has been realized that the normalization (2.8) is not innocuous for predicting counterfactual policy effects (see e.g. Norets and Tang, 2014; Arcidiacono and Miller, 2015; Aguirregabiria and Suzuki, 2014; Kalouptsidi, Scott, and Souza-Rodrigues, 2015). In the Supplemental Material, we show two things. First, the normalization (2.8) will bias the counterfactual policy predictions, if the per period utility of alternative $\mathtt{0}$ depends on the value of S_t. Second, the normalization of Assumption 4.(iii) will not bias the counterfactual policy predictions.

By assuming discrete state space (Assumption 4.(ii)), the per period utility functions $\mu_t^{0}(S_t)$ and $\mu_t^{1}(S_t)$, the CCP $p_t(S_t)$, the ASVF $v_t^{0}(S_t)$ and $v_t^{1}(S_t)$, and the ex ante value functions $v_t(S_t)$ are all finitely dimensional. Denote $\mu_t^{0} = (\mu_t^{0}(s_1), \ldots, \mu_t^{0}(s_{d_s}))^\intercal$, and μ_t^{1}, p_t, v_t^{0}, v_t^{1} and v_t are defined similarly. It should be remarked that our identification results below hold for any finite number of states d_s. Let $f_{t+1}^{\mathtt{d}}(S_{t+1}|S_t)$ be the conditional probability function of S_{t+1} given S_t and $D_t = \mathtt{d}$, and let $F_{t+1}^{\mathtt{d}}$ be the state transition matrix from S_t to S_{t+1} given choice $D_t = \mathtt{d}$. Denote $f_{t+1}^{\mathtt{1/0}}(S_{t+1}|S_t) \equiv f_{t+1}^{\mathtt{1}}(S_{t+1}|S_t) - f_{t+1}^{\mathtt{0}}(S_{t+1}|S_t)$ and $F_{t+1}^{\mathtt{1/0}} \equiv F_{t+1}^{\mathtt{1}} - F_{t+1}^{\mathtt{0}}$.

Example (Female labor force participation model)**.** Our particular model is based on Keane, Todd, and Wolpin (2011, section 3.1). In each year t, a married woman makes a labor force participation decision $D_t \in \{\mathtt{0}, \mathtt{1}\}$, where $\mathtt{1}$ is "to work" and $\mathtt{0}$ is "not to work", to maximize the expected remaining lifetime utility.

The per period utility depends on the household consumption (cons_t) and the number of young children (kid_t) in the household.[3] Consumption equals the household's income net of child-care expenditures. The household income is the sum of the husband's income (y_t) and the wife's income (wage_t) if she works. The per-child child-care cost is β if she works, and zero if she stays at home. So consumption is

$$\text{cons}_t = y_t + \text{wage}_t \cdot D_t - \beta \text{kid}_t \cdot D_t.$$

[3] We do not model the fertility decision, and assume the arrival of children as an exogenous stochastic process.

Suppose the wage offer function takes the following form

$$\text{wage}_t = \alpha_1 + \alpha_2 \text{xp}_t + \alpha_3 (\text{xp}_t)^2 + \alpha_4 \text{edu} + \omega_t,$$

where xp_t is the working experience (measured by the number of prior periods the woman has worked) of the woman in year t, edu is her education level, and ω_t is a random shock, which is independent of the wife's working experience and education. The wife's working experience xp_t evolves by

$$\text{xp}_{t+1} = \text{xp}_t + D_t.$$

Assume the per period utility functions associated with the two alternatives are

$$\begin{aligned} u_t^1(S_t, \varepsilon_t^1) &= \text{cons}_t + \varepsilon_t^1 \\ &= y_t + \alpha_1 + \alpha_2 \text{xp}_t + \alpha_3 (\text{xp}_t)^2 + \alpha_4 \text{edu} - \beta \text{kid}_t + \varepsilon_t^1, \qquad (2.9) \\ u_t^0(S_t, \varepsilon_t^0) &= \mu_t^0(y_t, \text{kid}_t) + \varepsilon_t^0. \end{aligned}$$

Besides the observable state variables about the woman, we also observe her husband's working experience xp_t^h and education level edu^h. Given husband's income y_t, xp_t^h and edu^h do not affect the per period utility but affect the husband's future income. These two state variables excluded from the per period utility function will be useful for identification of the structural parameters. Let $S_t = (y_t, \text{xp}_t, \text{edu}, \text{kid}_t, \text{xp}_t^h, \text{edu}^h)$ be the vector of observable state variables.

The problem is dynamic because the woman's current working decision D_t affects her working experience in the next period: $xp_{t+1} = \text{xp}_t + D_t$. As in the general model, the woman's choice D_t solves the Bellman equation (2.5) with the per period utility functions being substituted by equation (2.9).

We are interested in predicting the labor supply effects of some counterfactual intervention, such as child-care subsidy by the government or slower wage growth due to economic recession. In terms of the CCP, this means we would like to know the new CCP after imposing these counterfactual interventions. To answer these questions, we first need to identify and estimate the structural parameters.

2.2 Data and structural parameters of the model

Researchers only observe T *consecutive* decision periods, rather than the whole decision process. Denote the T sampling periods by $1, 2, \ldots, T$. It should be remarked that the first sampling period 1 does not need to be the first decision period, nor does the last sampling period T correspond to the terminal decision period T_*. Denote the data by $\mathsf{D} \equiv (D_1, S_1, \ldots, D_T, S_T)$, whose support is $\mathcal{D} \equiv (\{0, 1\} \times \mathcal{S})^T$. Let θ denote the vector of structural parameters of this model including per period utility functions $(\mu_t^0,\ \mu_t^1)$, discount factors (δ_t) and transition matrices $(F_t^0,\ F_t^1)$ in each period t. It will be useful to reparameterize (μ_t^0, μ_t^1) as $(\mu_t^0, \mu_t^{1/0})$, where $\mu_t^{1/0} \equiv (\mu_t^{1/0}(s_1), \ldots, \mu_t^{1/0}(s_{d_s}))^\top$ with $\mu_t^{1/0}(s) \equiv \mu_t^1(s) - \mu_t^0(s)$. Let $\theta_t \equiv (\mu_t^0, \mu_t^{1/0}, \delta_t, F_t^0, F_t^1)$ for $t = 1, \ldots, T-1$, and let $\theta_T \equiv (v_T^0, v_T^1, F_T^0, F_T^1)$ instead of $\theta_T \equiv (\mu_T^0, \mu_T^{1/0}, \delta_T, F_T^0, F_T^1)$, because the CCP $p_T(S_T)$ cannot be determined by the per period utility functions μ_T^0 and $\mu_T^{1/0}$ alone when $T < T_*$. Let $\theta \equiv (\theta_1, \ldots, \theta_T)$, and let Θ be the parameter space.

We consider identification for such data that we call a short panel not only because short panel data are common in empirical studies, but also because the number of time periods turns out to play an important role in the identification of DPDC models. As shown below, when the discount factors are known, one needs at least three consecutive periods to identify nonstationary DPDC models without the terminal conditions. In the presence of terminal conditions, we can identify the model with two consecutive periods data, when the discount factors are known. If the discount factors are unknown, we need one additional period data to identify the discount factors.

3 An example with four-period dynamic discrete choice

To develop some intuition for the general results in section 4 and 5, we consider a *four* period dynamic discrete choice model.

The goal is to show that with the Exclusion Restriction below, we can identify the per period utility functions without assuming that $\mu_t^0(s_1) = \cdots = \mu_t^0(s_{d_s}) = 0$, and the per period utility functions can be estimated by a closed form estimator. We maintain the following three assumptions in this section. First, assume that ε_t^0 and ε_t^1 are independent and follow the type-1 extreme value distribution (EVD). Second, the state transition matrices F_t^0 and F_t^1 are time invariant and denoted by F^0 and F^1, respectively. We also omit the subscript "t" in the conditional expectations E_t^0, E_t^1 and $\mathrm{E}_t^{1/0}$. Third, assume that the discount factor is constant over time and denoted by δ.

We will study three cases below. In case 1 (subsection 3.1), researchers observe the decisions in the last two decision periods. In case 2 (subsection 3.2), we have data of the first two decision periods. In the last case (subsection 3.3), researchers observe the decisions in the first *three* decision periods. Since period 4 is the terminal decision period, we have "terminal condition" in the first case, but not in the other two cases. The comparison between case 1 and 2 will show that "terminal conditions" do not help the identification of DPDC models. In the first two cases, we assume that the discount factor δ is known. In the third case, in which one additional period data are available, we identify the discount factor.

3.1 Identification and estimation with the data of the last two periods

In period 4 (terminal period), there is no continuation value for the choice. Hence the decision rule in period 4 is described by a logit model: $D_4 = 1(\varepsilon_4^{0} - \varepsilon_4^{1} < \mu_4^{1}(S_4) - \mu_4^{0}(S_4))$. The CCP $p_4(S_4)$ in period 4 is

$$p_4(S_4) = G(\mu_4^{1/0}(S_4)), \tag{3.1}$$

where $G(\cdot)$ is the logistic distribution function. It follows from the properties of the EVD that

$$\begin{aligned} v_4(S_4) &= \mathrm{E}_{\varepsilon_4}[V_4(S_4, \varepsilon_4)|S_4] \\ &= \mathrm{E}_{\varepsilon_4}\left[\max_{d\in\{0,1\}} \mu_4^{d}(S_4) + \varepsilon_4^{d} \middle| S_4\right] \\ &= \mu_4^{0}(S_4) + [\gamma - \ln(1 - p_4(S_4))] \\ &\equiv \mu_4^{0}(S_4) + \psi(p_4(S_4)) \end{aligned} \tag{3.2}$$

where $\gamma \approx 0.5772$ is Euler's constant. It follows from the CCP formula (2.7) that the CCP in period 3 is

$$\begin{aligned} p_3(S_3) &= G(\mu_3^{1/0}(S_3) + \delta\,\mathrm{E}^{1/0}[v_4(S_4)|S_3]) \\ &= G\left(\mu_3^{1/0}(S_3) + \delta\,\mathrm{E}^{1/0}[\mu_4^{0}(S_4)|S_3] + \delta\,\mathrm{E}^{1/0}[\psi(p_4(S_4))|S_3]\right). \end{aligned} \tag{3.3}$$

Let $\phi(p) \equiv \ln p - \ln(1-p)$ be the inverse of the logistic distribution function. It follows from equation (3.1) and (3.3) that

$$\phi(p_4(S_4)) = \mu_4^{1/0}(S_4), \tag{3.4a}$$

$$\phi(p_3(S_3)) = \mu_3^{1/0}(S_3) + \delta\,\mathrm{E}^{1/0}[\mu_4^{0}(S_4)|S_3] + \delta\,\mathrm{E}^{1/0}[\psi(p_4(S_4))|S_3]. \tag{3.4b}$$

From data (D_3, S_3, D_4, S_4), we can identify and estimate the state transition matrices F^{0} and F^{1}, and the CCP $p_3(S_3)$ and $p_4(S_4)$. The utility difference in the terminal period $\mu_4^{1/0}$ is then identified from equation (3.4a). However, $\mu_3^{1/0}$ and μ_4^{0} cannot be identified from equation (3.4b) without further restriction even when the discount factor δ is known, because both $\mu_3^{1/0}(S_3)$ and $\mathrm{E}^{1/0}[\mu_4^{0}(S_4)|S_3]$ are unknown functions of S_3.

When the discount factor δ is known, we will show how to identify and estimate $\mu_3^{1/0}$ and μ_4^0 using the following Exclusion Restriction. Note that $\mu_4^1 = \mu_4^0 + \mu_4^{1/0}$ is identified, when μ_4^0 and $\mu_4^{1/0}$ are identified.

Exclusion Restriction. *The vector of observable state variables S_t has two parts X_t and Z_t. Let $S_t = (X_t, Z_t)$, where $X_t \in \mathcal{X} \equiv \{x_1, \ldots, x_{d_x}\}$ and $Z_t \in \mathcal{Z} \equiv \{z_1, \ldots, z_{d_z}\}$. Assume that*

$$\mu_t^1(X_t, Z_t) = \mu_t^1(X_t) \qquad \text{and} \qquad \mu_t^0(X_t, Z_t) = \mu_t^0(X_t).$$

Assume that $\mathcal{S} = \mathcal{X} \times \mathcal{Z}$, so that $d_s = d_x \cdot d_z$. In particular, let

$$\mathcal{S} = \mathrm{vec}\begin{bmatrix} (x_1, z_1) & (x_2, z_1) & \ldots & (x_{d_x}, z_1) \\ \vdots & \vdots & \vdots & \vdots \\ (x_1, z_{d_z}) & (x_2, z_{d_z}) & \ldots & (x_{d_x}, z_{d_z}) \end{bmatrix}.$$

For $d_x = d_z = 2$, this means $\mathcal{S} = \{(x_1, z_1), (x_1, z_2), (x_2, z_1), (x_2, z_2)\}$.

Suppose $\mathcal{X} = \{x_1, x_2\}$ and $\mathcal{Z} = \{z_1, z_2\}$ in this section. Evaluating equation (3.4b) at each $(x_i, z_j) \in \mathcal{X} \times \mathcal{Z}$, we have

$$\phi(p_3(x_i, z_j)) = \mu_3^{1/0}(x_i) + \delta\, \mathrm{E}^{1/0}[\mu_4^0(X_4)|x_i, z_j] + \delta\, \mathrm{E}^{1/0}[\psi(p_4(S_4))|x_i, z_j], \tag{3.5}$$

for $i, j = 1, 2$. For each $x_i \in \mathcal{X}$, the difference $\phi(p_3(x_i, z_1)) - \phi(p_3(x_i, z_2))$ depends linearly on μ_4^0, but not on $\mu_3^{1/0}$. We are going to identify μ_4^0 using the differences $\phi(p_3(x_i, z_1)) - \phi(p_3(x_i, z_2))$, $i = 1, 2$. Then $\mu_3^{1/0}$ is identified by the above the display. For $i = 1, 2$, using the difference $\phi(p_3(x_i, z_1)) - \phi(p_3(x_i, z_2))$, we have

$$b_i = \mathrm{E}^{1/0}[\mu_4^0(X_4)|x_i, z_1] - \mathrm{E}^{1/0}[\mu_4^0(X_4)|x_i, z_2] \tag{3.6}$$

with b_i, $i = 1, 2$, being defined by

$$b_i \equiv \left(\delta^{-1}\phi(p_3(x_i, z_1)) - \mathrm{E}^{1/0}[\psi(p_4(S_4))|x_i, z_1]\right) \\ - \left(\delta^{-1}\phi(p_3(x_i, z_2)) - \mathrm{E}^{1/0}[\psi(p_4(S_4))|x_i, z_2]\right).$$

Equation (3.6) can be organized as the following linear system of equations,

$$A\mu_4^0 = b, \tag{3.7}$$

where $b \equiv (b_1, b_2)^\intercal$ and

$$A \equiv \begin{bmatrix} f^{1/0}(x_1|x_1, z_1) - f^{1/0}(x_1|x_1, z_2) & f^{1/0}(x_2|x_1, z_1) - f^{1/0}(x_2|x_1, z_2) \\ f^{1/0}(x_1|x_2, z_1) - f^{1/0}(x_1|x_2, z_2) & f^{1/0}(x_2|x_2, z_1) - f^{1/0}(x_2|x_2, z_2) \end{bmatrix}.$$

Using the notation $F^{1/0} = F^{1} - F^{0}$, the matrix A can be written alternatively as follows,

$$A = MF^{1/0}(I_2 \otimes 1_2),$$

where I_2 is the 2×2 identity matrix, "$\otimes$" is Kronecker product, $1_2 = (1,1)^\intercal$, and

$$M \equiv \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}. \tag{3.8}$$

The linear system of equations like equation (3.7) will be frequently encountered in the sequel. The matrix A will always depend only on the state transition matrices and the discount factors; the vector b will always depend only on the CCP and the discount factors. However, their explicit definitions will change with respect to different model specifications. Note that $A\,1_2 = 0_2$. Hence 1_2 is a non-zero eigenvector of matrix A associated with the eigenvalue 0. The matrix A cannot have full rank. If $\operatorname{rank} A = 1$, the solution set of equation (3.7) is $\{\, A^+ b + c \cdot 1_2 : c \in \mathbb{R} \,\}$, where A^+ is the Moore-Penrose pseudoinverse of A (see lemma A.1 for proof). So the solution for μ_4^0 is unique up to a constant that does not change with respect to states. Note that if $X_4 \perp\!\!\!\perp Z_3 | (X_3, D_3)$, both columns of A are zero, hence $\operatorname{rank} A = 0$. Though the solution for μ_4^0 is not unique, we have a unique solution for the utility difference $\mu_3^{1/0} = (\mu_3^{1/0}(x_1), \mu_3^{1/0}(x_2))^\intercal$. Let $\mu_4^0 = A^+ b + c \cdot 1_2$ be an arbitrary solution of equation (3.7), it follows from equation (3.5) that

$$\begin{aligned} \mu_3^{1/0}(x_i) &= \phi(p_3(x_i, z_j)) - \delta \begin{bmatrix} f^{1/0}(x_1|x_i, z_j) \\ f^{1/0}(x_2|x_i, z_j) \end{bmatrix}^\intercal \big(A^+ b + c \cdot 1_2\big) - \delta\, \mathrm{E}^{1/0}[\psi(p_4(S_4))|x_i, z_j]. \\ &= \phi(p_3(x_i, z_j)) - \delta \begin{bmatrix} f^{1/0}(x_1|x_i, z_j) \\ f^{1/0}(x_2|x_i, z_j) \end{bmatrix}^\intercal \big(A^+ b\big) - \delta\, \mathrm{E}^{1/0}[\psi(p_4(S_4))|x_i, z_j], \end{aligned}$$

for both $j = 1$ and 2. Note that the above display does not depend on the unknown constant c, so $\mu_3^{1/0}(x_i)$ is identified for $i = 1, 2$. It should be remarked that $\mu_3^{1/0}(x_i)$ is linear in the discount factor δ, and such linearity will be used to identify the discount factor in subsection 3.3.

The per period utility function $\mu_4^0 = (\mu_4^0(x_1), \mu_4^0(x_2))^\intercal$ is identified with the normalization $\mu_4^0(x_1) = 0$ (Assumption 4.(iii)). With such normalization, we can identify μ_4^0 as

$$\mu_4^0 = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} A^+ b.$$

To estimate $\mu_3^{1/0}$ and μ_4^0, we only need to estimate the difference between the state transition matrices F^0 and F^1, and the CCP $p_3(S_3)$ and $p_4(S_4)$, with which A and b are then estimated. The per period utility functions $\mu_3^{1/0}$ and μ_4^0 can be estimated by the above displays after substituting the unknowns with their estimates.

We end case 1 with several remarks about the identifying power of the parametric specification of the per period utility functions.

Remark 1 (Identification of discount factor using parametric specification, Exclusion Restriction and terminal conditions). Suppose

$$\mu_t^{1/0}(X_t) = \alpha_{t,0} + X_t^\intercal \alpha_{t,1} \qquad \text{and} \qquad \mu_t^0(X_t) = \beta_{t,0} + X_t^\intercal \beta_{t,1}. \tag{3.9}$$

Under the above specification, equation (3.5) becomes

$$\phi(p_3(S_3)) = \alpha_{3,0} + X_3^\intercal \alpha_{3,1} + \mathrm{E}^{1/0}\left(X_4^\intercal | X_3, Z_3\right)(\delta\beta_{4,1}) - \mathrm{E}^{1/0}\left[\psi(p_4(S_4))|X_3, Z_3\right]\delta.$$

Note that the intercept term $\beta_{4,0}$ disappears because $\mathrm{E}^{1/0}(\beta_{4,0}|X_3, Z_3) = 0$, and this corresponds to our earlier conclusion that the per period utility function μ_4^0 is identified up to a constant. It follows that $(\alpha_{3,0}, \alpha_{3,1}, \delta\beta_{4,1}, \delta)$ can be identified if X_3, $\mathrm{E}^{1/0}(X_4|X_3, Z_3)$ and $\mathrm{E}^{1/0}[\psi(p_4(S_4))|X_3, Z_3]$ are not linearly dependent.

Remark 2. In general the discount factor is not identifiable with two periods data even with the Exclusion Restriction. Without parametric specification about the per period utility functions, we have

$$\phi(p_3(X_3, Z_3)) = \mu_3^{1/0}(X_3) + \delta\,\mathrm{E}^{1/0}[\mu_4^0(X_4)|X_3, Z_3] + \delta\,\mathrm{E}^{1/0}[\psi(p_4(S_4))|X_3, Z_3].$$

Let $\mathcal{U}$ be the space of the per period utility function $\mu_4^0(X_4)$. The linear specification $\mu_t^0 = \beta_{t,0} + X_t^\intercal \beta_{t,1}$ in Remark 1 assumes that $\mathcal{U}$ is the set of all linear functions of X_4. The discount factor δ may not be identified, because $\delta\,\mathrm{E}^{1/0}[\mu_4^0(X_4)|X_3, Z_3]$ could be any function of (X_3, Z_3). If the equation of unknown function $g(X_4)$,

$$\mathrm{E}^{1/0}[g(X_4)|X_3, Z_3] = \mathrm{E}^{1/0}[\psi(p_4(X_4, Z_4))|X_3, Z_3],$$

has a solution in $\mathcal{U}$, the discount factor cannot be identified.[4] (In this particular case, there is always a solution because the CCP $p_4(S_4)$ in the terminal period depends only on X_4 by the Exclusion Restriction.) Suppose $g(X_4)$ is one solution, then let $\tilde{\mu}_4^0(X_4) \equiv \mu_4(X_4) - g(X_4)$. We have

$$\phi(p_3(X_3, Z_3)) = \mu_3^{1/0}(X_3) + (\delta + c)\,\mathrm{E}^{1/0}[\tilde{\mu}_4^0(X_4)|X_3, Z_3] + (\delta + c)\,\mathrm{E}^{1/0}[\psi(p_4(S_4))|X_3, Z_3],$$

[4]In Remark 1, in order to identify the discount factor δ, we require that $\mathrm{E}^{1/0}(X_4|X_3, Z_3)$ are $\mathrm{E}^{1/0}[\psi(p_4(S_4))|X_3, Z_3]$ are not linearly dependent. Note that this is equivalent to the condition here that the equation $\mathrm{E}^{1/0}[g(X_4)|X_3, Z_3] = \mathrm{E}^{1/0}[\psi(p_4(S_4))|X_3, Z_3]$ has no solution in $\mathcal{U}$, which is the set of all linear functions of X_4 given the linear specification in Remark 1.

for any c such that $0 < \delta + c < 1$, and the discount factor is not identified.

Remark 3. Without the Exclusion Restriction, the per period utility functions are not identifiable in general even with the linear specification (3.9) and the terminal conditions. Suppose there is no excluded variable Z_t, and the state variable $S_t = X_t$ is a scalar. Assume that X_t follows an autoregressive process,

$$X_t = \rho_0 + \rho_1^{\mathtt{d}} X_{t-1} + \omega_t, \qquad \mathtt{d} \in \{\mathtt{0}, \mathtt{1}\},$$

with $\mathrm{E}(\omega_t|X_{t-1}) = 0$. Let $\rho_1^{\mathtt{1/0}} = \rho_1^{\mathtt{1}} - \rho_1^{\mathtt{0}}$. Equation (3.5) becomes

$$\phi(p_3(X_3)) = \alpha_{3,0} + X_3(\alpha_{3,1} + \rho_1^{\mathtt{1/0}}\delta\beta_{4,1}) - \mathrm{E}^{\mathtt{1/0}}[\ln(1 - p_4(X_4))|X_3]\delta.$$

We can only identify $(\alpha_{3,1} + \rho_1^{\mathtt{1/0}}\delta\beta_{4,1})$ as a whole. However, if one is willing to assume that the per period utility functions are time invariant, so that $\alpha_{3,1} = \alpha_{4,1}$, we then can identify $\beta_{4,1}$ separately from the sum $(\alpha_{3,1} + \rho_1^{\mathtt{1/0}}\delta\beta_{4,1})$, because $\alpha_{3,1} = \alpha_{4,1}$, $\rho_1^{\mathtt{1/0}}$ and δ are identified ($\alpha_{4,1}$ is identified because $\mu_4^{\mathtt{1/0}}(S_4)$ is identified from equation (3.4a)). This observation will be generalized to Proposition S.2 in the Supplemental Material.

3.2 Identification and estimation with data of the first two periods

Suppose now researchers observe the decisions in the *first two* periods only, hence there is no terminal condition in this case. It follows from equation (2.7) that the CCP in period 2 and 1 are

$$p_2(S_2) = G(\mu_2^{\mathtt{1/0}}(S_2) + \delta\,\mathrm{E}^{\mathtt{1/0}}[v_3(S_3)|S_2]), \tag{3.10}$$

$$p_1(S_1) = G(\mu_1^{\mathtt{1/0}}(S_1) + \delta\,\mathrm{E}^{\mathtt{1/0}}[v_2(S_2)|S_1]). \tag{3.11}$$

It is similar to the derivations in equation (3.2) that we have

$$v_2(S_2) = \mathrm{E}_{\varepsilon_2}[V_3(S_3, \varepsilon_3)|S_2] = \mu_2^{\mathtt{0}}(S_2) + \delta\,\mathrm{E}^{\mathtt{0}}[v_3(S_3)|S_2] + \psi(p_2(S_2)). \tag{3.12}$$

Similar to equation (3.4), we have the following equations from equation (3.10)-(3.12),

$$\phi(p_1(S_1)) = \mu_1^{\mathtt{1/0}}(S_1) + \delta\,\mathrm{E}^{\mathtt{1/0}}[v_2(S_2)|S_1], \tag{3.13a}$$

$$\phi(p_2(S_2)) = \mu_2^{\mathtt{1/0}}(S_2) + \delta\,\mathrm{E}^{\mathtt{1/0}}[v_3(S_3)|S_2], \tag{3.13b}$$

$$v_2(S_2) = \mu_2^{\mathtt{0}}(S_2) + \delta\,\mathrm{E}^{\mathtt{0}}[v_3(S_3)|S_2] + \psi(p_2(S_2)). \tag{3.13c}$$

Without terminal condition, the per period utility functions difference $\mu_2^{\mathtt{1/0}}$ cannot be identified from equation (3.13b). As in case 1 of subsection 3.1, the other parameters, $\mu_1^{\mathtt{1/0}}$, $\mu_2^{\mathtt{0}}$ and δ, are not identified. So we conclude that the terminal condition only helps identify the difference between the per period utility functions in the terminal period.

Applying the Exclusion Restriction, equation (3.13) becomes

$$\phi(p_1(x_i, z_j)) = \mu_1^{1/0}(x_i) + \delta\, \mathrm{E}^{1/0}[v_2(S_2)|x_i, z_j], \qquad i, j = 1, 2, \quad (3.14a)$$

$$\phi(p_2(x_i, z_j)) = \mu_2^{1/0}(x_i) + \delta\, \mathrm{E}^{1/0}[v_3(S_3)|x_i, z_j], \qquad i, j = 1, 2, \quad (3.14b)$$

$$v_2(x_i, z_j) = \mu_2^{0}(x_i) + \delta\, \mathrm{E}^{0}[v_3(S_3)|x_i, z_j] + \psi(p_2(x_i, z_j)), \qquad i, j = 1, 2. \quad (3.14c)$$

We want to identify $\mu_1^{1/0}$, $\mu_2^{1/0}$ and μ_2^0 by solving $\left(\mu_1^{1/0}, \mu_2^{1/0}, \mu_2^0, v_2, v_3\right)$ explicitly from equation (3.14b). Note that the per period utility function μ_1^0 does not appear in the above equations, hence cannot be identified. We solve equation (3.14b) by following the steps below.

Step 1: Eliminate $\mu_1^{1/0}$, $\mu_2^{1/0}$ and μ_2^0 from equation (3.14b). Let $\phi_t(i, j) \equiv \phi(p_t(x_i, z_j))$, $\psi_t(i, j) \equiv \psi(p_t(x_i, z_j))$ and $\bar{v}_3(S_3) \equiv \delta v_3(S_3)$. We have the following,

$$\begin{aligned}
\delta^{-1}(\phi_1(i,1) - \phi_1(i,2)) &= \mathrm{E}^{1/0}[v_2(S_2)|x_i, z_1] - \mathrm{E}^{1/0}[v_2(S_2)|x_i, z_2], \\
\phi_2(i,1) - \phi_2(i,2) &= \mathrm{E}^{1/0}[\bar{v}_3(S_3)|x_i, z_1] - \mathrm{E}^{1/0}[\bar{v}_3(S_3)|x_i, z_2], \\
\psi_2(i,1) - \psi_2(i,2) &= v_2(x_i, z_1) - v_2(x_i, z_2) - \mathrm{E}^{0}[\bar{v}_3(S_3)|x_i, z_1] + \mathrm{E}^{0}[\bar{v}_3(S_3)|x_i, z_2],
\end{aligned}$$

for $i = 1, 2$. When the discount factor δ is known, the above system is equivalent to

$$A \begin{bmatrix} v_2 \\ \bar{v}_3 \end{bmatrix} = b_2, \qquad (3.15)$$

where the unknown is

$$\begin{bmatrix} v_2 \\ \bar{v}_3 \end{bmatrix} = \mathrm{vec} \begin{bmatrix} v_2(x_1, z_1) & v_2(x_2, z_1) & \bar{v}_3(x_1, z_1) & \bar{v}_3(x_2, z_1) \\ v_2(x_1, z_2) & v_2(x_2, z_2) & \bar{v}_3(x_1, z_2) & \bar{v}_3(x_2, z_2) \end{bmatrix},$$

the coefficient matrix A is a 6×8 matrix,

$$A \equiv \begin{bmatrix} MF^{1/0} & 0 \\ 0 & MF^{1/0} \\ M & -MF^{0} \end{bmatrix},$$

with the M matrix as defined by equation (3.8), and b_2 is a 6-dimensional vector,

$$b_2 \equiv \mathrm{vec} \begin{bmatrix} (\phi_1(1,1) - \phi_1(1,2))/\delta & \phi_2(1,1) - \phi_2(1,2) & \psi_2(1,1) - \psi_2(1,2) \\ (\phi_1(2,1) - \phi_1(2,2))/\delta & \phi_2(2,1) - \phi_2(2,2) & \psi_2(2,1) - \psi_2(2,2) \end{bmatrix}.$$

Step 2: Solve v_2 and $\bar{v}_3$ from equation (3.15). Let A^{+} be the Moore-Penrose pseudoinverse of matrix A, then

$$\begin{bmatrix} v_2^+ \\ \bar{v}_3^+ \end{bmatrix} \equiv A^+ b_2$$

solves equation (3.14b). Because we need to use v_t^+ and $\bar{v}_{t+1}^+$ separately, it is useful to split the matrix A^+ into two parts:

$$A^+ = \begin{bmatrix} A_u^+ \\ A_l^+ \end{bmatrix},$$

where A_u^+ and A_l^+ are the 4×6 matrices formed by the first and last 4 rows of matrix A^+, respectively. Then

$$\begin{bmatrix} v_2^+ \\ \bar{v}_3^+ \end{bmatrix} = \begin{bmatrix} A_u^+ b_2 \\ A_l^+ b_2 \end{bmatrix}.$$

If $\operatorname{rank} A = 6$, we know from lemma A.2 that the solution set of equation (3.15) is that

$$\left\{ \begin{bmatrix} v_2^+ + c_2 \cdot 1_4 \\ \bar{v}_3^+ + c_3 \cdot 1_4 \end{bmatrix} : c_2, c_3 \in \mathbb{R} \right\}.$$

Step 3: Identify the per period utility functions $\mu_1^{1/0}$, $\mu_2^{1/0}$ and μ_2^0. Suppose $\operatorname{rank} A = 6$, and let $v_2 = v_2^+ + c_2 \cdot 1_4$ and $\bar{v}_3 = \bar{v}_3^+ + c_3 \cdot 1_4$ be arbitrary solutions of equation (3.15). Let

$$f^{1/0}(i,j) \equiv (f^{1/0}(x_1, z_1|x_i, z_j), f^{1/0}(x_1, z_2|x_i, z_j), f^{1/0}(x_2, z_1|x_i, z_j), f^{1/0}(x_2, z_2|x_i, z_j))^\intercal.$$

Then associated with v_2 and $\bar{v}_3$, we have the following from equation (3.14b): for $j = 1, 2$,

$$\mu_1^{1/0}(x_i) = \phi(p_1(x_i, z_j)) - \delta f^{1/0}(i,j)^\intercal A_u^+ b_2, \tag{3.16}$$

$$\mu_2^{1/0}(x_i) = \phi(p_2(x_i, z_j)) - f^{1/0}(i,j)^\intercal A_l^+ b_2, \tag{3.17}$$

$$\mu_2^0(x_i) = v_2^+(x_i, z_j) - f^{1/0}(i,j)^\intercal A_l^+ b_2 - \psi(p_2(x_i, z_j)) + (c_2 - c_3).$$

The constant $c_2 - c_3$ in $\mu_2^0(x_i)$ can be determined by the normalization condition that $\mu_2^0(x_1) = 0$. So we conclude that the per period utility functions $\mu_1^{1/0}$, $\mu_2^{1/0}$ and μ_2^0 are identified.

Given the explicit formulas for the per period utility functions, their estimation is easy. We again estimate the CCP and the state transition matrices first, then plug their estimates into the above formulas to estimate the per period utility functions.

3.3 Identification of the discount factor with three-period data

Suppose we have data $(D_1, S_1, D_2, S_2, D_3, S_3)$. Applying the identification arguments of case 2 (subsection 3.2) with data (D_1, S_1, D_2, S_2) and data (D_2, S_2, D_3, S_3), respectively, we will have two formulas for $\mu_2^{1/0}(x_i)$:

$$\mu_2^{1/0}(x_i) = \phi(p_2(x_i, z_j)) - \delta f^{1/0}(i,j)^\intercal A_u^+ b_3, \tag{3.18}$$

$$\mu_2^{1/0}(x_i) = \phi(p_2(x_i, z_j)) - f^{1/0}(i,j)^\intercal A_l^+ b_2, \tag{3.19}$$

where equation (3.18) follows from equation (3.16) with data (D_2, S_2, D_3, S_3), and equation (3.19) follows from (3.17) with data (D_1, S_1, D_2, S_2). Equation (3.18) and (3.19) give the following equation,

$$\delta f^{1/0}(i,j)^\intercal A_u^+ b_3 = f^{1/0}(i,j)^\intercal A_l^+ b_2, \tag{3.20}$$

about the discount factor δ. In the Supplemental Material, we derived the solution of the discount factor δ. We list the conclusion here:

$$\delta = \frac{r_{2,u}(2,j)(r_{2,l}(1,j) - r_{3,u}(1,j)) - r_{2,u}(1,j)(r_{2,l}(2,j) - r_{3,u}(2,j))}{r_{3,l}(1,j)r_{2,u}(2,j) - r_{3,l}(2,j)r_{2,u}(1,j)}, \tag{3.21}$$

where

$$r_{3,l}(i,j) \equiv h_u(i,j)b_{3,l}, \qquad r_{3,u}(i,j) \equiv h_u(i,j)b_{3,u},$$

$$r_{2,u}(i,j) \equiv h_l(i,j)b_{2,u}, \qquad r_{2,l}(i,j) \equiv h_l(i,j)b_{2,l},$$

$$h_u(i,j) \equiv f^{1/0}(i,j)^\intercal A_u^+, \qquad h_l(i,j) \equiv f^{1/0}(i,j)^\intercal A_l^+,$$

$$b_{t,u} \equiv \text{vec} \begin{bmatrix} \phi_{t-1}(1,1) - \phi_{t-1}(1,2) & 0 & 0 \\ \phi_{t-1}(2,1) - \phi_{t-1}(2,2) & 0 & 0 \end{bmatrix},$$

$$b_{t,l} \equiv \text{vec} \begin{bmatrix} 0 & \phi_t(1,1) - \phi_t(1,2) & \psi_t(1,1) - \psi_t(1,2) \\ 0 & \phi_t(2,1) - \phi_t(2,2) & \psi_t(2,1) - \psi_t(2,2) \end{bmatrix}.$$

According to equation (3.21), it is necessary that the denominator $\tilde{r} \equiv r_{3,l}(1,j)r_{2,u}(2,j) - r_{3,l}(2,j)r_{2,u}(1,j) \neq 0$. In the Supplemental Material, we show that to ensure $\tilde{\gamma} \neq 0$, it is necessary that (i) the choice D_t changes the state transition distributions; (ii) the state variable X_t should affect the difference between the state transition distributions under the two alternatives given the excluded variable Z_t; (iii) the excluded variable Z_t should still change the CCP conditional on X_t.

4 Identification of structural parameters

We first show that the identification of DPDC models is equivalent to the identification of a linear GMM model. Then, applying this equivalence, we prove a list of identification results. At the end, we consider the extension to allow for unobservable heterogeneity.

4.1 Linear GMM representation of DPDC models

Our DPDC model maps its structural parameters θ to a joint probability function $f(\mathsf{D};\theta)$ of data $\mathsf{D} \equiv (D_1, S_1, \ldots, D_T, S_T)$. The structural parameters θ are identified if for any $f(\mathsf{D}) \in \{\, f(\mathsf{D};\theta) : \theta \in \Theta \,\}$, the system of equations

$$f(\mathsf{D}) = f(\mathsf{D};\theta),\ \forall \mathsf{D} \in \mathcal{D}, \tag{4.1}$$

has a unique solution for θ in the parameter space Θ.

Let $\vec{S}_t \equiv (S_1, \ldots, S_{t-1})^\intercal$ and $\vec{D}_t \equiv (D_1, \ldots, D_{t-1})^\intercal$. We can always write

$$f(\mathsf{D}) = f(S_1)\mathrm{P}(D_1|S_1)\prod_{t=2}^{T} f_t(S_t|\vec{S}_t, \vec{D}_t)\mathrm{P}(D_t|S_t, \vec{S}_t, \vec{D}_t). \tag{4.2}$$

By the Markovian assumptions 2 and 3, we have $f_t(S_t|\vec{S}_t, \vec{D}_t) = f_t(S_t|S_{t-1}, D_{t-1})$ and $\mathrm{P}(D_t|\vec{S}_t, \vec{D}_t) = \mathrm{P}(D_t|S_t)$. So equation (4.2) equals the following,

$$f(\mathsf{D}) = f_1(S_1)\mathrm{P}(D_1|S_1)\prod_{t=2}^{T} \mathrm{P}(D_t|S_t) f_t(S_t|S_{t-1}, D_{t-1}),$$

where $\mathrm{P}(D_t|S_t) = (p_t(S_t))^{D_t}(1 - p_t(S_t))^{1-D_t}$. Similarly, we can decompose $f(\mathsf{D};\theta)$ by

$$f(\mathsf{D};\theta) = f_1(S_1;\theta)\mathrm{P}(D_1|S_1;\theta)\prod_{t=2}^{T} \mathrm{P}(D_t|S_t;\theta) f_t(S_t|S_{t-1}, D_{t-1};\theta),$$

where $\mathrm{P}(D_t|S_t;\theta) = (p_t(S_t;\theta))^{D_t}(1 - p_t(S_t;\theta))^{1-D_t}$. Because of the above decomposition of $f(\mathsf{D})$ and $f(\mathsf{D};\theta)$, it can be verified that equation (4.1) is equivalent to the following[5]

$$f_1(S_1) = f_1(S_1;\theta), \tag{4.3a}$$

$$f_{t+1}(S_{t+1}|S_t, D_t) = f_{t+1}(S_{t+1}|S_t, D_t;\theta),\ t = 1, \ldots, T-1, \tag{4.3b}$$

$$p_t(S_t) = p_t(S_t;\theta),\ t = 1, \ldots, T. \tag{4.3c}$$

We conclude from equation (4.3a) and (4.3b) that the state transition probabilities are identified. In the remainder of the identification analysis, we assume that the state transition probabilities are known and focus on the identification of per period utility functions (μ_t^{0} and $\mu_t^{\mathsf{1/0}}$) and discount factors (δ_t).

The attention now is equation (4.3c), which requires the explicit form of the CCP $p_t(S_t;\theta)$ in terms of the structural parameters θ. It follows from the CCP formula of equation (2.7) that

$$p_t(S_t;\theta) = G(\mu_t^{\mathsf{1/0}}(S_t) + \delta_t\,\mathrm{E}_{t+1}^{\mathsf{1/0}}[v_{t+1}(S_{t+1})|S_t]),\ t = 1, \ldots, T-1, \tag{4.4}$$

$$p_T(S_T;\theta) = G(v_T^{\mathsf{1}}(S_T) - v_T^{\mathsf{0}}(S_T)), \tag{4.5}$$

where G is the CDF of $\tilde{\varepsilon} \equiv \varepsilon_t^{\mathsf{0}} - \varepsilon_t^{\mathsf{1}}$. Because the CDF G is known and strictly increasing (Assumption 3.(i)), its inverse $\phi(\cdot) \equiv G^{-1}(\cdot)$ is known. So that

$$\phi(p_t(S_t;\theta)) = v_t^{\mathsf{1}}(S_t) - v_t^{\mathsf{0}}(S_t),\ t = 1, \ldots, T. \tag{4.6}$$

It should be noted that the ex ante value functions $\{\, v_{t+1} : t = 1, \ldots, T-1 \,\}$ in equation (4.4) are not structural parameters. So we express $v_t(S_t)$ in terms of the structural parameters. It follows from the definition of $v_t(S_t)$ in equation (2.2) and

[5] Take $T = 2$ for example, so that $\mathsf{D} = (D_1, S_1, D_2, S_2)$. If equation (4.3) holds, we clearly have $f(\mathsf{D}) = f(\mathsf{D};\theta)$. Suppose $f(\mathsf{D}) = f(\mathsf{D};\theta)$, and we will show equation (4.3). We first have $f_1(S_1) = \sum_{D_1,D_2,S_2} f(\mathsf{D})$ and $f_1(S_1;\theta) = \sum_{D_1,D_2,S_2} f(\mathsf{D};\theta)$. From $f(\mathsf{D}) = f(\mathsf{D};\theta)$, we conclude $f_1(S_1) = f_1(S_1;\theta)$. The notation $\sum_{D_1,D_2,S_2}$ means sum over all values of (D_1, D_2, S_2) in their support. We next have $f(S_1, D_1) = \sum_{D_2,S_2} f(\mathsf{D})$ and $f(S_1, D_1;\theta) = \sum_{D_2,S_2} f(\mathsf{D};\theta)$, hence $f(S_1, D_1) = f(S_1, D_1;\theta)$. Because $f(S_1, D_1) = f_1(S_1)\mathrm{P}(D_1|S_1)$, $f(S_1, D_1;\theta) = f_1(S_1;\theta)\mathrm{P}(D_1|S_1;\theta)$ and $f_1(S_1) = f_1(S_1;\theta)$, we conclude $\mathrm{P}(D_1|S_1) = \mathrm{P}(D_1|S_1;\theta)$, which is equivalent to $p_1(S_1) = p_1(S_1;\theta)$. Following the same strategy, we conclude $f(S_1, D_1, S_2) = f(S_1, D_1, S_2;\theta)$, which implies that $f_2(S_2|S_1, D_1) = f_2(S_2|S_1, D_1;\theta)$ as $f(S_1, D_1) = f(S_1, D_1;\theta)$. We conclude $p_2(S_2) = p_2(S_2;\theta)$ by $f(\mathsf{D}) = f(\mathsf{D};\theta)$ and $f(S_1, D_1, S_2) = f(S_1, D_1, S_2;\theta)$.

the Bellman equation (2.5) that

$$
\begin{aligned}
v_t(S_t) &= \int \max\{v_t^0(S_t) + \varepsilon_t^0, v_t^1(S_t) + \varepsilon_t^1\} \,\mathrm{d}F(\varepsilon_t^0, \varepsilon_t^1) \\
&= v_t^0(S_t) + \int \max\{\varepsilon_t^0, v_t^1(S_t) - v_t^0(S_t) + \varepsilon_t^1\} \,\mathrm{d}F(\varepsilon_t^0, \varepsilon_t^1) \\
&= v_t^0(S_t) + \left[\mathrm{E}(\varepsilon_t^0) + \int \max\{0, \phi(p_t(S_t;\theta)) - \tilde{\varepsilon}_t\} \,\mathrm{d}G(\tilde{\varepsilon}_t)\right] \\
&= v_t^0(S_t) + \psi(p_t(S_t;\theta)),
\end{aligned}
$$

where ψ depends only on the CDF G of the utility shocks $(\varepsilon_t^0, \varepsilon_t^1)^\intercal$. Replacing v_t^0 in the above display with its definition in equation (2.4), we have

$$
\begin{aligned}
v_t(S_t) &= \mu_t^0(S_t) + \delta_t \,\mathrm{E}_{t+1}^0[v_{t+1}(S_{t+1})|S_t] + \psi(p_t(S_t;\theta)), \quad t < T, \\
v_T(S_T) &= v_T^0(S_T) + \psi(p_T(S_T;\theta)).
\end{aligned} \tag{4.7}
$$

Note that $v_T^0 \in \theta_T$ and $p_T(S_T;\theta) = G(v_T^1(S_T) - v_T^0(S_T))$ are determined by θ_T. So that $v_T(S_T)$ is completely determined by θ_T.

Substituting $p_t(S_t;\theta)$ in equation (4.3c) with equation (4.4) and (4.5), we have the following

$$
\begin{aligned}
p_t(S_t) &= G(\mu_t^{1/0}(S_t) + \delta_t \,\mathrm{E}_{t+1}^{1/0}[v_{t+1}(S_{t+1})|S_t]), \ t = 1, \ldots, T-1, \\
p_T(S_T) &= G(v_T^1(S_T) - v_T^0(S_T)), \\
v_t(S_t) &= \mu_t^0(S_t) + \delta_t \,\mathrm{E}_{t+1}^0[v_{t+1}(S_{t+1})|S_t] + \psi(p_t(S_t;\theta)), \ t = 2, \ldots, T-1, \\
v_T(S_T) &= v_T^0(S_T) + \psi(p_T(S_T;\theta)).
\end{aligned}
$$

In this system of equations, the *known* objects are the CCP $\{\, p_t(S_t) : t = 1, \ldots, T \,\}$ and the state transition matrices hidden in the conditional expectation operators $\mathrm{E}_{t+1}^{1/0}(\cdot|S_t)$ and $\mathrm{E}_{t+1}^0(\cdot|S_t)$; the *unknowns* are per period utility functions $\{\mu_1^{1/0}, \ldots, \mu_{T-1}^{1/0}, \mu_2^0, \ldots, \mu_{T-1}^0\}$, ex ante value functions $\{v_2, \ldots, v_T\}$, the two ASVF v_T^0 and v_T^1, and the discount factors $\{\delta_1, \ldots, \delta_{T-1}\}$. One component of the structural parameters is identified iff the above system of equations has a unique solution for it.

Two remarks help to simplify the above system of equations. First, using the invertibility of the CDF G and the identities $p_t(S_t;\theta) = p_t(S_t)$, the above system has the same solutions as

$$\phi(p_t(S_t)) = \mu_t^{1/0}(S_t) + \delta_t \mathrm{E}_{t+1}^{1/0}[v_{t+1}(S_{t+1})|S_t],\ t = 1, \ldots, T-1, \tag{4.8a}$$

$$\phi(p_T(S_T)) = v_T^1(S_T) - v_T^0(S_T), \tag{4.8b}$$

$$v_t(S_t) = \mu_t^0(S_t) + \delta_t \mathrm{E}_{t+1}^0[v_{t+1}(S_{t+1})|S_t] + \psi(p_t(S_t)),\ t = 2, \ldots, T-1, \tag{4.8c}$$

$$v_T(S_T) = v_T^0(S_T) + \psi(p_T(S_T)). \tag{4.8d}$$

Second, equation (4.8b) and (4.8d) state that v_T^0 and v_T^1 are uniquely determined by v_T. Hence, in order to solve for $(\theta_1, \ldots, \theta_T)$ from equation (4.8), we can solve for $\theta_1, \ldots, \theta_{T-1}$ and v_T. Moreover, the solutions of $(\theta_1, \ldots, \theta_{T-1}, v_T)$, which appears only in equation (4.8a) and (4.8c), do not depend on equation (4.8b) and (4.8d). So equation (4.8) has the same solution for $(\theta_1, \ldots, \theta_{T-1}, v_T)$ as the following system,

$$\begin{cases} \phi(p_t(s_t)) = \mu_t^{1/0}(s_t) + \delta_t \mathrm{E}_{t+1}^{1/0}[v_{t+1}(S_{t+1})|s_t], & t = 1, \ldots, T-1, \\ \psi(p_t(s_t)) = v_t(s_t) - \mu_t^0(s_t) - \delta_t \mathrm{E}_{t+1}^0[v_{t+1}(S_{t+1})|s_t], & t = 2, \ldots, T-1. \end{cases} \tag{ID}$$

The identification analysis below will be based on checking if there is a unique solution for (some parts of) $(\theta_1, \ldots, \theta_{T-1}, v_T)$ by solving $(\theta_1, \ldots, \theta_{T-1}, v_T)$ from equation (ID).

Equation (ID) has the feature that given the discount factors δ_t, it is linear in all the other unknowns; meanwhile, given the other unknowns, equation (ID) is linear in the discount factors. When the discount factors are known, the uniqueness of solution is very easy to check because equation (ID) is linear in all the other unknowns. More explicitly, using the notation of F_{t+1}^0 and $F_{t+1}^{1/0}$, equation (ID) can be written as follows,

$$\begin{cases} \phi(p_t) = \mu_t^{1/0} + \delta_t F_{t+1}^{1/0} v_{t+1}, & t = 1, \ldots, T-1, \\ \psi(p_t) = v_t - \mu_t^0 - \delta_t F_{t+1}^0 v_{t+1}, & t = 2, \ldots, T-1, \end{cases} \tag{ID'}$$

where

$$\phi(p_t) \equiv (\phi(p_t(s_1)), \ldots, \phi(p_t(s_{d_s})))^\intercal \qquad \text{and} \qquad \psi(p_t) \equiv (\psi(p_t(s_1)), \ldots, \psi(p_t(s_{d_s})))^\intercal.$$

In this sense, we claim that the identification of DPDC models is equivalent to identification of a linear GMM system, henceforth a familiar problem. The necessary condition for identification is that the number of equations is greater than the number of unknowns (order condition). If the order condition fails, we shall consider restrictions that can eliminate certain number of unknowns, or add more equations by increasing the number of time periods T in panel data.

4.2 Identification of DPDC models by the linear GMM representation

A sequence of identification results will be derived by using the linear GMM representation of the DPDC model in equation (ID). The unknowns in equation (ID) are $\{ \mu_1^{1/0}, \ldots, \mu_{T-1}^{1/0}, \mu_2^{0}, \ldots, \mu_{T-1}^{0}, v_2, \ldots, v_T, \delta_1, \ldots, \delta_{T-1} \}$. Without restriction, equation (ID) has $(2T-3)\cdot d_s$ equations with $(3T-4)\cdot d_s+(T-1)$ unknowns. This implies that the structural parameters are not identified even when all discount factors are known (removing $T-1$ unknowns). The non-identification of the DPDC model has long been known in the literature (Rust, 1994; Magnac and Thesmar, 2002). The problem of interests is what restrictions shall we use? We focus on the identification using the Exclusion Restriction stated in section 3.

Under the Exclusion Restriction, $\mu_t^{\mathsf{d}} = (\mu_t^{\mathsf{d}}(x_1), \ldots, \mu_t^{\mathsf{d}}(x_{d_x}))^\intercal$ is a d_x-dimensional vector. The Exclusion Restriction is satisfied in the female labor force participation example, where $S_t = (y_t, \mathrm{xp}_t, \mathrm{edu}, \mathrm{kid}_t, \mathrm{xp}_t^h, \mathrm{edu}^h)$ with $X_t = (y_t, \mathrm{xp}_t, \mathrm{edu}, \mathrm{kid}_t)$ and $Z_t = (\mathrm{xp}_t^h, \mathrm{edu}^h)$. In general, given a set of state variables X_t that affect per period utilities, one searches for Z_t by looking for the variables that affect X_{t+1} but not affect per period utilities given X_t. For example, in Rust's (1987) bus engine replacement application, X_t is the mileage of the bus. Then Z_t could be characteristics of the bus' route, which will affect the bus' mileage in the next period, but not the current maintenance cost given the mileage.

We have shown the identification power of the Exclusion Restriction in the previous section. Below, we provide more general identification results. Applying the Exclusion Restriction, equation (ID) becomes

$$\begin{cases} \phi(p_t(X_t, Z_t)) = \mu_t^{1/0}(X_t) + \delta_t \, \mathrm{E}_{t+1}^{1/0}[v_{t+1}(S_{t+1})|X_t, Z_t], & t = 1, \ldots, T-1, \\ \psi(p_t(X_t, Z_t)) = v_t(X_t, Z_t) - \mu_t^{0}(X_t) - \delta_t \, \mathrm{E}_{t+1}^{0}[v_{t+1}(S_{t+1})|X_t, Z_t], & t = 2, \ldots, T-1. \end{cases} \tag{4.9}$$

There are $(2T-3)\cdot d_x + (T-1)\cdot d_s + (T-1)$ unknowns and $(2T-3)\cdot d_s$ equations. When the discount factors are known (removing $T-1$ unknowns), $d_z \geq 3$ and $T \geq 3$, we have more equations than unknowns. It should be remarked that when $T < 3$, the order condition always fails regardless of the value of d_z. When $T = 2$, we have only

$$\phi(p_1(X_1, Z_1)) = \mu_1^{1/0}(X_1) + \delta_1 \, \mathrm{E}_2^{1/0}[v_2(S_2)|X_1, Z_1].$$

In general, we do not have

$$\phi(p_2(X_2, Z_2)) = \mu_2^{1/0}(X_2) + \delta_2 \operatorname{E}_3^{1/0}[v_3(S_3)|X_2, Z_2],$$
$$\psi(p_2(X_2, Z_2)) = v_2(X_2, Z_2) - \mu_2^{0}(X_2) - \delta_2 \operatorname{E}_3^{0}[v_3(S_3)|X_2, Z_2],$$

because the state transition matrices F_3^0 and F_3^1 are unknown given only data (D_1, S_1, D_2, S_2). However, if one assumes that the state transition matrices are time invariant as we did in section 3, we can use the above two equations.

We first focus on the identification with known discount factors. Let

$$\bar{v}_{t+1}(S_{t+1}) \equiv \delta_t v_{t+1}(S_{t+1})$$

be the discounted ex ante value function. For each period $t = 2, \dots, T-1$, we can solve the unknowns $(\mu_{t-1}^{1/0}, \mu_t^{1/0}, \mu_t^0, v_t, v_{t+1})$ from the following part of equation (4.9),

$$\begin{cases} \phi(p_{t-1}(X_{t-1}, Z_{t-1})) = \mu_{t-1}^{1/0}(X_{t-1}) + \delta_{t-1} \operatorname{E}_t^{1/0}[v_t(S_t)|X_{t-1}, Z_{t-1}], \\ \phi(p_t(X_t, Z_t)) = \mu_t^{1/0}(X_t) + \operatorname{E}_{t+1}^{1/0}[\bar{v}_{t+1}(S_{t+1})|X_t, Z_t], \\ \psi(p_t(X_t, Z_t)) = v_t(X_t, Z_t) - \mu_t^0(X_t) - \operatorname{E}_{t+1}^{0}[\bar{v}_{t+1}(S_{t+1})|X_t, Z_t]. \end{cases} \tag{4.10}$$

Ranging period t from 2 to $T-1$, all unknowns $\mu_1^{1/0}, \dots, \mu_{T-1}^{1/0}, \mu_2^0, \dots, \mu_{T-1}^0, v_2, \dots, v_T$ will then be solved. Equation (4.10) is very similar to equation (3.13), for which we have shown the detailed solution steps. To save space, we give only the solutions of $(\mu_{t-1}^{1/0}, \mu_t^{1/0}, \mu_t^0)$ and leave the solution details in the Supplemental Material. Define

$$A_t \equiv \begin{bmatrix} MF_t^{1/0} & 0 \\ 0 & MF_{t+1}^{1/0} \\ M & -MF_{t+1}^0 \end{bmatrix} \qquad \text{and} \qquad b_t \equiv \begin{bmatrix} \delta_{t-1}^{-1} M\phi(p_{t-1}) \\ M\phi(p_t) \\ M\psi(p_t) \end{bmatrix}. \tag{4.11}$$

If $\operatorname{rank} A_t = 2\cdot(d_s - 1)$, we have unique solution of $\mu_t^{1/0}$, μ_t^0 and $\mu_{t-1}^{1/0}$ from equation (4.9)

$$\mu_t^{1/0} = W(\phi(p_t) - F_{t+1}^{1/0} A_{t,l}^+ b_t), \tag{4.12}$$
$$\mu_{t-1}^{1/0} = W(\phi(p_{t-1}) - \delta_{t-1} F_t^{1/0} A_{t,u}^+ b_t), \tag{4.13}$$
$$\mu_t^0 = WL(A_{t,u}^+ b_t - F_{t+1}^0 A_{t,l}^+ b_t - \psi(p_t)),$$

where $W \equiv I_{d_x} \otimes (d_z^{-1} \cdot 1_{d_z})^\intercal$, $A_{t,u}^+$ and $A_{t,l}^+$ are the $d_s \times [3 \cdot d_x \cdot (d_z - 1)]$ matrices formed by the first and last d_s rows of the Moore-Penrose pseudoinverse matrix A_t^+ of matrix A_t, respectively,

$$M \equiv I_{d_x} \otimes \begin{bmatrix} 1 & -1 & & \\ & \ddots & \ddots & \\ & & 1 & -1 \end{bmatrix}_{(d_z-1)\times d_z} \quad \text{and} \quad L \equiv \left[\begin{array}{c|ccc} 0 & 0 & \dots & 0 \\ \hline -1 & 1 & & \\ \vdots & & \ddots & \\ -1 & & & 1 \end{array}\right]_{d_s \times d_s}, \tag{4.14}$$

In Appendix A, we derive the closed-form solution for $(\mu_1^{1/0}, \dots, \mu_{T-1}^{1/0})$ and $(\mu_2^0, \dots, \mu_{T-1}^0)$ from equation (4.9).

Proposition 1 (Identification with the Exclusion Restriction, known discount factors and $T \geq 3$)**.** *In addition to Assumptions 1-4, suppose the Exclusion Restriction holds, the discount factors are known and $T \geq 3$. For $t = 2, \dots, T-1$, let the matrix A_t be defined by equation (4.11). If $\operatorname{rank} A_t = 2 \cdot (d_s - 1)$, then the per period utility functions $\mu_t^{1/0}$, μ_t^0 and $\mu_{t-1}^{1/0}$ are identified. Define*

$$\mu_{1:(T-1)}^{1/0} \equiv ((\mu_1^{1/0})^\intercal, \dots, (\mu_{T-1}^{1/0})^\intercal)^\intercal, \qquad \mu_{2:(T-1)}^{0} \equiv ((\mu_2^{0})^\intercal, \dots, (\mu_{T-1}^{0})^\intercal)^\intercal,$$

$$\phi(p_{1:(T-1)}) \equiv ((\phi(p_1))^\intercal, \dots, (\phi(p_{T-1}))^\intercal)^\intercal, \quad \psi(p_{2:(T-1)}) \equiv ((\psi(p_2))^\intercal, \dots, (\psi(p_{T-1}))^\intercal)^\intercal.$$

We have

$$\mu_{1:(T-1)}^{1/0} = (I_{T-1} \otimes W)\,[\phi(p_{1:(T-1)}) - (\tilde{\Lambda}^{-1} \otimes I_{d_s}) F_{2:T}^{1/0} A_{1:T}^{+} b_{1:T}], \tag{4.15}$$

$$\mu_{2:(T-1)}^{0} = [I_{T-2} \otimes (WL)][(\Lambda^{-1} \otimes I_{d_s}) \check{F}_{3:T}^{0} A_{1:T}^{+} b_{1:T} - \psi(p_{2:(T-1)})]. \tag{4.16}$$

where $\Lambda \equiv \operatorname{diag}\left(\delta_1, \prod_{r=1}^{2} \delta_r, \cdots, \prod_{r=1}^{T-2} \delta_r\right)$, $\tilde{\Lambda} \equiv \operatorname{diag}\left(1, \delta_1, \prod_{r=1}^{2} \delta_r, \cdots, \prod_{r=1}^{T-2} \delta_r\right)$,

$$A_{1:T} \equiv \begin{bmatrix} (I_{T-1} \otimes M)\, F_{2:T}^{1/0} \\ (I_{T-2} \otimes M)\, \check{F}_{3:T}^{0} \end{bmatrix}, \qquad b_{1:T} \equiv \begin{bmatrix} (I_{T-1} \otimes M)(\tilde{\Lambda} \otimes I_{T-1})\phi(p_{1:(T-1)}) \\ (I_{T-2} \otimes M)(\Lambda \otimes I_{T-2})\psi(p_{2:(T-1)}) \end{bmatrix},$$

$$F_{2:T}^{1/0} \equiv \begin{bmatrix} F_2^{1/0} & & & \\ & F_3^{1/0} & & \\ & & \ddots & \\ & & & F_T^{1/0} \end{bmatrix}, \quad \text{and} \quad \check{F}_{3:T}^{0} \equiv \begin{bmatrix} I_{d_s} & -F_3^0 & & & \\ & I_{d_s} & -F_4^0 & & \\ & & \ddots & \ddots & \\ & & & I_{d_s} & -F_T^0 \end{bmatrix}.$$

Proof. See Appendix A. □

When the panel data have the number of time periods greater than 4, we can also identify the discount factors using the strategy of subsection 3.3. From equation (4.12) and (4.13), we have two formulas of $\mu_t^{1/0}$,

$$\mu_t^{1/0} = W(\phi(p_t) - F_{t+1}^{1/0} A_{t,l}^+ b_t),$$
$$\mu_t^{1/0} = W(\phi(p_t) - \delta_t F_{t+1}^{1/0} A_{t+1,u}^+ b_{t+1}),$$

using data $(D_{t-1}, S_{t-1}, \ldots, D_{t+1}, S_{t+1})$ and $(D_t, S_t, \ldots, D_{t+2}, S_{t+2})$, respectively. Equalizing the above two formulas, we have an equation of the discount factors δ_{t-1} and δ_t, which are hidden in b_t and b_{t+1}:

$$WF_{t+1}^{1/0} A_{t,l}^+ b_t - \delta_t W F_{t+1}^{1/0} A_{t+1,u}^+ b_{t+1} = 0. \tag{4.17}$$

We derive the explicit solutions of (δ_{t-1}, δ_t) below. Define

$$b_{t,u} \equiv \operatorname{vec}\begin{bmatrix} M\phi(p_{t-1}) & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad b_{t,l} \equiv \operatorname{vec}\begin{bmatrix} 0 & M\phi(p_t) & M\psi(p_t) \end{bmatrix},$$

so that $b_t = \delta_{t-1}^{-1} b_{t,u} + b_{t,l}$. Let

$$H_{t,l} \equiv WF_{t+1}^{1/0} A_{t,l}^+ \qquad \text{and} \qquad H_{t+1,u} \equiv WF_{t+1}^{1/0} A_{t+1,u}^+.$$

Then equation (4.17) is written as follows,

$$\tilde{R}_t \begin{bmatrix} \delta_{t-1}^{-1} \\ -\delta_t \end{bmatrix} = (H_{t,l} b_{t,l} - H_{t+1,u} b_{t+1,u}),$$

where

$$\tilde{R}_t \equiv \begin{bmatrix} H_{t,l} b_{t,u} & H_{t+1,u} b_{t+1,l} \end{bmatrix}. \tag{4.18}$$

If rank $\tilde{R}_t = 2$, we have the unique solution of $(\delta_{t-1}^{-1}, -\delta_t)^\intercal = \tilde{R}_t^+ (H_{t,l} b_{t,l} - H_{t+1,u} b_{t+1,u})$.

Proposition 2 (Identification of discount factors with the Exclusion Restriction and $T \geq 4$)**.** *Suppose the conditions of Proposition 1 hold and $T \geq 4$. For each $t = 2, \ldots, T-1$, if the matrix $\tilde{R}_t$ defined in equation* (4.18) *has full rank, the discount factors δ_{t-1} and δ_t are identified.*

When the agent's dynamic programming problem is stationary[6], and the Exclusion Restriction holds, equation (ID) becomes

$$\begin{cases} \phi(p(X,Z)) = \mu^{1/0}(X) + \delta \, \mathrm{E}^{1/0}[v(X', Z')|X, Z], \\ \psi(p(X,Z)) = v(X,Z) - \mu^0(X) - \delta \, \mathrm{E}^0[v(X', Z')|X, Z]. \end{cases} \tag{4.19}$$

The linear system of equations (4.19) has $2d_s$ equations with $2d_x + d_s$ unknowns. So if $d_s \geq 2d_x$, we may be able to identify the structural parameters. In particular,

[6]By "stationary dynamic programming problem", we mean that the decision horizon T_* is infinite, and the per period utility functions, the discount factors and the state transition distributions are time invariant.

when $d_s = d_x \cdot d_z$, the order condition $d_s \geq 2d_x$ would be satisfied as long as $d_z \geq 2$. The identification of the per period utility functions $(\mu^{\mathtt{1/0}}, \mu^{\mathtt{0}})$ will be based on solving $(\mu^{\mathtt{1/0}}, \mu^{\mathtt{0}})$ explicitly from equation (4.19). The solution details are left in the Supplemental Material.

Proposition 3 (Identification with the Exclusion Restriction, known discount factors and stationarity)**.** *In addition to Assumptions 1-4, suppose the Exclusion Restriction holds, the discount factors are known and that the agent's dynamic programming problem is stationary. Let the matrix A and the vector b be defined by*

$$A \equiv \begin{bmatrix} \delta M F^{\mathtt{1/0}} \\ M(I - \delta F^{\mathtt{0}}) \end{bmatrix} \qquad \textit{and} \qquad b \equiv \begin{bmatrix} M\phi(p) \\ M\psi(p) \end{bmatrix}.$$

If $T \geq 2$ and $\operatorname{rank} A = d_s - 1$, the per period utility functions $\mu^{\mathtt{1/0}}$ and $\mu^{\mathtt{0}}$ are identified. Moreover, we have

$$\mu^{\mathtt{1/0}} = W(\phi(p) - \delta F^{\mathtt{1/0}} A^{+} b),$$
$$\mu^{\mathtt{0}} = WL[(I - \delta F^{\mathtt{0}}) A^{+} b - \psi(p)].$$

In the Supplemental Material, we have two other identification results (Proposition S.1 and S.2). Proposition S.1 shows that when the excluded variables are time invariant, such as husband's education level, the per period utility functions are identified and satisfy the formulas in Proposition 1 if $\operatorname{rank} A_t = 2d_s - d_z - 1$ for $t = 2, \ldots, T-1$. When there are no excluded variables Z_t, an alternative way to identify per period utility functions is to assume that the per period utility functions are time invariant but the state transition matrices are time varying. Then time itself is an excluded variable. Proposition S.2 shows that when there are at least 4 sampling periods in data, the time invariant per period utility function can be identified with explicit formulas.

Remark 4 (Extension to multinomial choices). Suppose the choice set is $\{\mathtt{0}, \mathtt{1}, \ldots, \mathtt{J}\}$. By the Hotz-Miller's inversion formula (Hotz and Miller, 1993), there exists $\{\phi^{\mathtt{j}} : \mathtt{j} = \mathtt{1}, \ldots, \mathtt{J}\}$ and ψ such that

$$\begin{aligned} v_t^{\mathtt{j}}(S_t) - v_t^{\mathtt{0}}(S_t) &= \phi^{\mathtt{j}}(p_t(S_t)) \\ v_t(S_t) - v_t^{\mathtt{0}}(S_t) &= \psi(p_t(S_t)) \end{aligned},$$

where $p_t(S_t) \equiv (\mathrm{P}(D_t = \mathtt{1}|S_t), \ldots, \mathrm{P}(D_t = \mathtt{J}|S_t))^{\intercal}$. Equation (ID) becomes

$$\begin{cases} \phi^{\mathtt{j}}(p_t(S_t)) = \mu_t^{\mathtt{j/0}}(S_t) + \delta_t \operatorname{E}_{t+1}^{\mathtt{j/0}}[v_{t+1}(S_{t+1})|S_t], & t = 1, \ldots, T-1, \ \mathtt{j} = \mathtt{1}, \ldots, \mathtt{J}, \\ \psi(p_t(S_t)) = v_t(S_t) - \mu_t^{\mathtt{0}}(S_t) - \delta_t \operatorname{E}_{t+1}^{\mathtt{0}}[v_{t+1}(S_{t+1})|S_t], & t = 2, \ldots, T-1. \end{cases}$$

Each alternative $\mathtt{j}$ contributes $d_s \cdot (T-1)$ equations (associated with $\{ \phi^{\mathtt{j}}(p_t(S_t)) : t = 1, \ldots, T-1 \}$); meanwhile the alternative $\mathtt{j}$ brings $d_s \cdot (T-1)$ additional unknowns $\{ \mu_t^{\mathtt{j}/\mathtt{0}} : t = 1, \ldots, T-1 \}$. So the degree of underidentification does not change as we include more alternatives. However, with the Exclusion Restriction, we have

$$\begin{cases} \phi^{\mathtt{j}}(p_t(S_t)) = \mu_t^{\mathtt{j}/\mathtt{0}}(X_t) + \delta_t \operatorname{E}_{t+1}^{\mathtt{j}/\mathtt{0}}[v_{t+1}(S_{t+1})|S_t], & t = 1, \ldots, T-1,\ \mathtt{j} = \mathtt{1}, \ldots, \mathtt{J}, \\ \psi(p_t(S_t)) = v_t(S_t) - \mu_t^{\mathtt{0}}(X_t) - \delta_t \operatorname{E}_{t+1}^{\mathtt{0}}[v_{t+1}(S_{t+1})|S_t], & t = 2, \ldots, T-1. \end{cases}$$

Each alternative $\mathtt{j}$ still contributes $d_s \cdot (T-1)$ new equations; meanwhile the alternative $\mathtt{j}$ brings only $d_x \cdot (T-1)$ additional unknowns $\{ \mu_t^{\mathtt{j}/\mathtt{0}} : t = 1, \ldots, T-1 \}$, because $\mu_t^{\mathtt{j}/\mathtt{0}}$ is now d_x-dimensional vector. So more alternatives provide more information about the structural parameters. The exact identification results for multinomial choices are slightly different from the above propositions, but the general idea is similar.

4.3 Extension to DPDC models with fixed effect

We will show that the linear GMM system can be used to identify dynamic discrete choice models with fixed effect. We focus on the dynamic discrete choice models with finite horizon T_*. Also, assume that the discount factor $\delta_t = \delta$ is time invariant. An agent now has unobservable fixed effect η. Let $\Omega_t \equiv (S_t, \eta_t, \varepsilon_t^{\mathtt{0}}, \varepsilon_t^{\mathtt{1}})$ be the state variables at time t. We consider the following per period utility functions

$$u_t(\Omega_t, D_t) = D_t \cdot (\mu_t^{\mathtt{1}}(S_t) + \eta + \varepsilon_t^{\mathtt{1}}) + (1 - D_t) \cdot (\mu_t^{\mathtt{0}}(S_t) + \varepsilon_t^{\mathtt{0}}).$$

So the per period utility difference is $\mu_t^{\mathtt{1}/\mathtt{0}}(S_t) + \eta + \varepsilon_t^{\mathtt{1}} - \varepsilon_t^{\mathtt{0}}$, which is additive in η. We allow η to be correlated with the observable state variables S_t. The utility shocks $(\varepsilon_t^{\mathtt{0}}, \varepsilon_t^{\mathtt{1}})$ still satisfy Assumption 3 and 4. We need the following assumptions.

Assumption 5. (i) *For each t, $S_{t+1} \perp\!\!\!\perp \eta | (S_t, D_t)$.*

(ii) *For each t, $\eta \perp\!\!\!\perp (\varepsilon_t^{\mathtt{0}}, \varepsilon_t^{\mathtt{1}})$.*

Let $v_t(S_t, \eta) = \operatorname{E}_{(\varepsilon_t^{\mathtt{0}}, \varepsilon_t^{\mathtt{1}})}[V_t(\Omega_t)|S_t, \eta]$ be the ex ante value function, and let $v_t^{\mathtt{1}}(S_t, \eta)$ and $v_t^{\mathtt{0}}(S_t, \eta)$ be the ASVF. Using the feature that the fixed effect η enters in the utility functions additively, it can be shown by solving the dynamic programming problem backwardly that $v_t(S_t, \eta)$, $v_t^{\mathtt{1}}(S_t, \eta)$ and $v_t^{\mathtt{0}}(S_t, \eta)$ are additive in η. Letting

$$g_t(\eta) = \left(\frac{1 - \delta^{T_* - t}}{1 - \delta} \right) \eta,$$

we have

$$
\begin{aligned}
v_t(S_t,\eta) &= v_t^*(S_t) + g_t(\eta),\\
v_t^{\mathtt{1}}(S_t,\eta) &= v_t^{\mathtt{1}*}(S_t) + \eta + (g_t(\eta)-\eta),\\
v_t^{\mathtt{0}}(S_t,\eta) &= v_t^{\mathtt{0}*}(S_t) + (g_t(\eta)-\eta),\\
v_t^{\mathtt{d}*}(S_t) &= \mu_t^{\mathtt{d}}(S_t) + \delta\,\mathrm{E}_{t+1}^{\mathtt{d}}[v_{t+1}^*(S_{t+1})|S_t], \qquad \mathtt{d}=\mathtt{0},\mathtt{1},\\
v_t^*(S_t) &= v_t^{\mathtt{0}*}(S_t) + \int \max(0, v_t^{\mathtt{1}*}(S_t) - v_t^{\mathtt{0}*}(S_t) - \tilde{\varepsilon}_t)\,\mathrm{d}G(\tilde{\varepsilon}_t).
\end{aligned}
$$

Here $v_t^*(S_t), v_t^{\mathtt{0}*}(S_t)$ and $v_t^{\mathtt{1}*}(S_t)$ are functions of S_t only. Our goal is to find an equation that is similar to equation (ID), so we can use the exclusion restriction to identify the per period utility functions and discount factor. To this end, we need a control variable W_t such that $S_t \perp\!\!\!\perp \eta|W_t$.

Assumption 6. (i) *There is a control W_t such that $S_t \perp\!\!\!\perp \eta|W_t$ and $W_t \perp\!\!\!\perp \tilde{\varepsilon}_t$. Assume that W_t is uniformly distributed between 0 and 1.*

(ii) *Letting $\xi_t \equiv \tilde{\varepsilon}_t - \eta$, the distribution function $F(\xi_t)$ is known and is strictly increasing.*

The uniformly distributed control variable can be motivated by Imbens and Newey (2009). The known distribution of ξ_t is a strong assumption, for which we make a remark at the end.

Given the control variable W_t, we have

$$
\begin{aligned}
p_t(S_t,W_t) &= \mathrm{E}[D_t|S_t,W_t]\\
&= \mathrm{E}[\tilde{\varepsilon}_t - \eta \le v_t^{\mathtt{1}*}(S_t) - v_t^{\mathtt{0}*}(S_t)|S_t,W_t]\\
&= \int_{-\infty}^{v_t^{\mathtt{1}*}(S_t)-v_t^{\mathtt{0}*}(S_t)} \mathrm{d}F(\xi_t|W_t). \qquad (4.20)
\end{aligned}
$$

Letting $q_t(S_t) = \mathrm{E}_w[p_t(S_t,W_t)] = \int_0^1 p_t(S_t,w)\,\mathrm{d}w$, we have

$$
q_t(S_t) = \int_{-\infty}^{v_t^{\mathtt{1}*}(S_t)-v_t^{\mathtt{0}*}(S_t)} \mathrm{d}F(\xi_t).
$$

Obviously, $q_t(S_t)$ is known from data. Since $F(\xi_t)$ is known and invertible, letting $\varphi(\cdot)$ be the inverse of $F(\xi_t)$, we have

$$
\begin{aligned}
\varphi(q_t(S_t)) &= v_t^{\mathtt{1}*}(S_t) - v_t^{\mathtt{0}*}(S_t) \qquad (4.21)\\
&= \mu_t^{\mathtt{1/0}}(S_t) + \delta\,\mathrm{E}_{t+1}^{\mathtt{1/0}}[v_{t+1}^*(S_{t+1})|S_t].
\end{aligned}
$$

Next, we have

$$\begin{aligned}
v_t^*(S_t) &= v_t^0(S_t) + \int \max(0, v_t^{1*}(S_t) - v_t^{0*}(S_t) - \tilde{\varepsilon}_t)\,\mathrm{d}G(\tilde{\varepsilon}_t) \\
&= v_t^0(S_t) + \int \max(0, \varphi(q_t(S_t)) - \tilde{\varepsilon}_t)\,\mathrm{d}G(\tilde{\varepsilon}_t) \\
&\equiv v_t^0(S_t) + \kappa(q_t(S_t)),
\end{aligned}$$

where $\kappa(q_t(s_t))$ is defined by the equation. The functional form of $\kappa(\cdot)$ depends on the known distribution functions $G(\tilde{\varepsilon}_t)$ and $F(\xi_t)$. In summary, we have the following two equations for each period t:

$$\begin{aligned}
\varphi(q_t(S_t)) &= \mu_t^{1/0}(S_t) + \delta\,\mathrm{E}_{t+1}^{1/0}[v_{t+1}^*(S_{t+1})|S_t], \\
\kappa(q_t(S_t)) &= v_t^*(S_t) - \mu_t^0(S_t) - \delta\,\mathrm{E}_{t+1}^0[v_{t+1}^*(S_{t+1})|S_t],
\end{aligned}$$

which has the same structure like equation (ID). So the previous arguments can be applied to identify $\mu_t^{1/0}(S_t)$, $\mu_t^0(S_t)$, δ and $v_t^*(S_t)$.

Remark 5. If we observe the choice in the terminal period T_*, and assume $\mu_{T_*}^{1/0}(S_{T_*}) = S_{T_*}^{\intercal}\alpha_{T_*}$, we then can identify the distribution function $F(\xi_t)$ by using the approach in Blundell and Powell (2004). In the terminal period, $v_{T_*}^{1*}(S_{T_*}) - v_{T_*}^{0*}(S_{T_*}) = \mu_{T_*}^{1/0}(S_{T*}) = S_{T_*}^{\intercal}\alpha_{T_*}$. For each pair of (s_{T_*}, s'_{T_*}) such that $q_{T_*}(s_{T_*}) = q_{T_*}(s'_{T_*})$, we have

$$\left(s_{T_*} - s'_{T_*}\right)^{\intercal}\alpha_{T_*} = 0.$$

If we have enough number of such pairs, we can identify α_{T*}, hence the distribution function $F(\xi_t)$.

Remark 6. If the vector of state variables S_t contains a continuous variable S_{1t}, with which $p_t(S_t, W_t)$ and $q_t(S_t)$ are differentiable, the conditional distribution function $F(\eta|W_t)$ is identifiable. Suppose $F(\xi_t|W_t)$ has density function $f(\xi_t|W_t)$, it follows from equation (4.20) and (4.21) that

$$p_t(S_t, W_t) = \int_{-\infty}^{\varphi(q_t(S_t))} f(\xi|W_t)\,\mathrm{d}\xi.$$

We then have

$$f\left(\xi = \varphi\left(q_t(S_t)\right)\middle|\, W_t\right) = \frac{\partial p_t(S_t, W_t)/\partial S_{1t}}{\partial\varphi\left(q_t\left(S_t\right)\right)/\partial S_{1t}}.$$

If the range of $\varphi(q_t(S_t))$ covers the support of ξ, $f(\xi|W_t)$ is identified. Because $\xi_t = \tilde{\varepsilon}_t - \eta$, $\tilde{\varepsilon}_t \perp\!\!\!\perp \eta$, and the distribution function of $\tilde{\varepsilon}_t$ is known, we identify $f(\eta|W_t)$.

5 Estimation

All identification results in the previous section are constructive and follow from the linear system of equations (ID). The solution of the linear system has a closed form. Therefore, it is natural to estimate these identified structural parameters by replacing population parameters by sample estimates of the closed form solutions. The estimation proceeds in two steps. In the first step, we estimate the CCP $\{p_t(S_t) : t = 1, \ldots, T-1\}$ and the transition matrix $\{F_{t+1}^{\mathtt{d}} : t = 1, \ldots, T-1, \mathtt{d} = \mathtt{0}, \mathtt{1}\}$. Let $\hat{p}_t(S_t)$ and $\hat{F}_{t+1}^{\mathtt{d}}$ be the estimates of the CCP $p_t(S_t)$ and transition matrix $F_{t+1}^{\mathtt{d}}$ for each alternative $\mathtt{d}$ and each period t. For small state space $\mathcal{S}$, the estimator of the CCP $p_t(S_t)$ is simply the proportion of $D_t = \mathtt{1}$ in data for each S_t. When the support of S_t is large, a kernel estimator of $p_t(S_t) = \mathrm{E}(D_t|S_t)$ might be preferable. Similarly, for small state space $\mathcal{S}$, an estimator of $F_{t+1}^{\mathtt{d}}$ is simply the empirical frequency table of the transitions from S_t to S_{t+1} given $D_t = \mathtt{d}$. When the support of S_t is large, a smoothed approach may be preferable to avoid the issue of empty cells; see Aitchison and Aitken (1976). The second step is to estimate the structural parameters using the closed form solutions of the linear system under different identifying restrictions.

We focus on the case with the Exclusion Restriction and known discount factors (Proposition 1). Moreover, assume that the transition matrices are also known.

5.1 Estimation of stationary DPDC models

For a stationary DPDC model with known discount factor and state transition matrices, we need only cross-sectional data to estimate the per period utility functions $\mu^{\mathtt{1}/\mathtt{0}}$ and $\mu^{\mathtt{0}}$. Let $\{ d_i, s_i : i = 1, \ldots, n \}$ be n agents' discrete choices and the corresponding states. For stationary DPDC models, it follows from Proposition 3 that

$$\mu^{\mathtt{1}/\mathtt{0}} = W(\phi(p) - \delta F^{\mathtt{1}/\mathtt{0}} A^{+} b),$$
$$\mu^{\mathtt{0}} = WL[(I - \delta F^{\mathtt{0}}) A^{+} b - \psi(p)].$$

Let $\hat{p}$ be the estimator of the CCP $p = (p(s_1), \ldots, p(s_{d_s}))^{\intercal}$, and let

$$\sqrt{n}(\hat{p} - p) \to_d \mathcal{N}(0, \Pi).$$

We then have the estimators $\hat{\mu}^{\mathtt{1}/\mathtt{0}}$ and $\hat{\mu}^{\mathtt{0}}$:

$$\hat{\mu}^{\mathtt{1}/\mathtt{0}} = W(\phi(\hat{p}) - \delta F^{\mathtt{1}/\mathtt{0}} A^{+} \hat{b}), \tag{5.1}$$
$$\hat{\mu}^{\mathtt{0}} = WL[(I - \delta F^{\mathtt{0}}) A^{+} \hat{b} - \psi(\hat{p})], \tag{5.2}$$

where

$$\hat{b} = \begin{bmatrix} M\phi(\hat{p}) \\ M\psi(\hat{p}) \end{bmatrix}.$$

It is easy to verify that

$$\sqrt{n}(\hat{\mu}^{1/0} - \mu^{1/0}) \to_d \mathcal{N}(0, (G^{1/0})\Pi(G^{1/0})^\intercal),$$
$$\sqrt{n}(\hat{\mu}^{0} - \mu^{0}) \to_d \mathcal{N}(0, (G^{0})\Pi(G^{0})^\intercal).$$

Here $G^{1/0}$ and G^0 are the gradient of $\mu^{1/0}$ and μ^0 with respect to p, respectively:

$$G^{1/0} \equiv \frac{\partial \mu^{1/0}}{\partial p} = W\nabla\phi(p) - WF^{1/0}A^{+}\nabla b, \qquad G^{0} \equiv \frac{\partial \mu^{0}}{\partial p} = WL(I - \delta F^0)A^{+}\nabla b - WL\nabla\psi(p),$$

where

$$\nabla\phi(p) \equiv \operatorname{diag}\left(\frac{\partial\phi(p(s_1))}{\partial p(s_1)}, \cdots, \frac{\partial\phi(p(s_{d_s}))}{\partial p(s_{d_s})}\right),$$
$$\nabla\psi(p) \equiv \operatorname{diag}\left(\frac{\partial\psi(p(s_1))}{\partial p(s_1)}, \cdots, \frac{\partial\psi(p(s_{d_s}))}{\partial p(s_{d_s})}\right),$$
$$\nabla b \equiv \begin{bmatrix} M\nabla\phi(p) \\ M\nabla\psi(p) \end{bmatrix}.$$

Though parametric specification of the per period utility functions is not necessary for our estimator, it is easy to incorporate the parametric specification into our estimators. Suppose each state x_i corresponds to a vector $\tilde{x}_i$ and

$$\mu^{1/0}(x_i) = \tilde{x}_i^\intercal \alpha \qquad \text{and} \qquad \mu^{0}(x_i) = \tilde{x}_i^\intercal \beta.$$

Then

$$\mu^{1/0} = \begin{bmatrix} \tilde{x}_1^\intercal \alpha \\ \vdots \\ \tilde{x}_{d_x}^\intercal \alpha \end{bmatrix} = \tilde{X}\alpha \qquad \text{and} \qquad \mu^{0} = \begin{bmatrix} \tilde{x}_1^\intercal \beta \\ \vdots \\ \tilde{x}_{d_x}^\intercal \beta \end{bmatrix} = \tilde{X}\beta.$$

In the numerical example of section 6, $\tilde{x}_i \equiv (1, x_i, x_i^2)^\intercal$ and $\mu^{1/0}(x_i) = \alpha_1 + \alpha_2 x_i + \alpha_3 x_i^2$.

Given the estimators $\hat{\mu}^{1/0}$ and $\hat{\mu}^0$ in equation (5.1) and (5.2), respectively, we can estimate α and β by

$$\hat{\alpha} = (\tilde{X}^\intercal \tilde{X})^{-1} \tilde{X}^\intercal \hat{\mu}^{1/0} \qquad \text{and} \qquad \hat{\beta} = (\tilde{X}^\intercal \tilde{X})^{-1} \tilde{X}^\intercal \hat{\mu}^{0}, \tag{5.3}$$

and

$$\sqrt{n}(\hat{\alpha}-\alpha) \to_d \mathcal{N}\Big(0, (\tilde{X}^\intercal \tilde{X})^{-1} \tilde{X}^\intercal (G^{1/0}) \mathit{\Pi} (G^{1/0})^\intercal \tilde{X} (\tilde{X}^\intercal \tilde{X})^{-1}\Big),$$
$$\sqrt{n}(\hat{\beta}-\beta) \to_d \mathcal{N}\Big(0, (\tilde{X}^\intercal \tilde{X})^{-1} \tilde{X}^\intercal (G^{0}) \mathit{\Pi} (G^{0})^\intercal \tilde{X} (\tilde{X}^\intercal \tilde{X})^{-1}\Big).$$

5.2 Estimation of nonstationary DPDC models

When the DPDC model is not stationary, we need at least two periods data when the discount factor and state transition matrices are known. Let $\{ d_{it}, s_{it} : i = 1, \ldots, n, t = 1, \ldots, T \}$ be n agents' discrete choices and the corresponding states over the T *sampling* periods. The estimation will be based on the formulas in Proposition 1. For any $t < s$, let $p_{t:s}^\intercal \equiv (p_t^\intercal, \ldots, p_s^\intercal)$, and let let $\hat{p}_{t:s}$ be its estimator.

We have the following estimator the per period utility functions:

$$\hat{\mu}^{1/0}_{1:(T-1)} = (I_{T-1} \otimes W) \left(\phi(\hat{p}_{1:(T-1)}) - (\tilde{\mathit{\Lambda}}^{-1} \otimes I_{d_s}) F^{1/0}_{2:T} A^{+}_{1:T} \hat{b}_{1:T} \right), \tag{5.4}$$

$$\hat{\mu}^{0}_{2:(T-1)} = [I_{T-2} \otimes (WL)] \left((\mathit{\Lambda}^{-1} \otimes I_{d_s}) \check{F}^{0}_{3:T} A^{+}_{1:T} \hat{b}_{1:T} - \psi(\hat{p}_{2:(T-1)}) \right). \tag{5.5}$$

where

$$\hat{b}_{1:T} \equiv \begin{bmatrix} (I_{T-1} \otimes M)(I_{T-1} \otimes \tilde{\mathit{\Lambda}}) \phi(\hat{p}_{1:(T-1)}) \\ (I_{T-2} \otimes M)(I_{T-2} \otimes \mathit{\Lambda}) \psi(\hat{p}_{2:(T-1)}) \end{bmatrix}.$$

Given

$$\sqrt{n} \left(\hat{p}_{1:(T-1)} - p_{1:(T-1)} \right) \to_d \mathcal{N}(0, \mathit{\Pi}).$$

we have

$$\sqrt{n} \left(\hat{\mu}^{1/0}_{1:(T-1)} - \mu^{1/0}_{1:(T-1)} \right) \to_d \mathcal{N}(0, (G^{1/0}_{1:T}) \mathit{\Pi} (G^{1/0}_{1:T})^\intercal),$$
$$\sqrt{n} \left(\hat{\mu}^{0}_{2:(T-1)} - \mu_{2:(T-1)} \right) \to_d \mathcal{N}(0, (G^{0}_{1:T}) \mathit{\Pi} (G^{0}_{1:T})^\intercal).$$

Here

$$
\begin{aligned}
G^{1/0}_{1:T} &\equiv \frac{\partial \mu^{1/0}_{1:(T-1)}}{\partial p_{1:(T-1)}} \\
&= (I_{T-1} \otimes W) \Big(\operatorname{diag}\left(\nabla\phi(p_1), \ldots, \nabla\phi(p_{T-1})\right) - (\tilde{\Lambda}^{-1} \otimes I_{d_s}) F^{1/0}_{2:T} A^{+}_{1:T} \nabla b_{1:T} \Big),
\end{aligned}
$$

$$
\begin{aligned}
G^{0}_{1:T} &\equiv \frac{\partial \mu^{0}_{2:(T-1)}}{\partial p_{1:(T-1)}} \\
&= [I_{T-2} \otimes (WL)] \left((\Lambda^{-1} \otimes I_{d_s}) \check{F}^{0}_{3:T} A^{+}_{1:T} \nabla b_{1:T} - \begin{bmatrix} 0 & \nabla\psi(p_2) & & \\ \vdots & & \ddots & \\ 0 & & & \nabla\psi(p_{T-1}) \end{bmatrix} \right),
\end{aligned}
$$

$$
\nabla b_{1:T} = \begin{bmatrix} (I_{T-1} \otimes M)(\tilde{\Lambda} \otimes I_{T-1}) \operatorname{diag}\left(\nabla\phi(p_1), \ldots, \nabla\phi(p_{T-1})\right) \\ \\ (I_{T-2} \otimes M)(\Lambda \otimes I_{T-2}) \begin{bmatrix} 0 & \nabla\psi(p_2) & & \\ \vdots & & \ddots & \\ 0 & & & \nabla\psi(p_{T-1}) \end{bmatrix} \end{bmatrix}.
$$

When there is parametric specification of the per period utility functions, we estimate the unknown parameters involved in the specification by the minimum distance principle. In particular, suppose each state x_i corresponds to a vector $\tilde{x}_i$, and suppose

$$
\mu^{1/0}_t(x_i) = \tilde{x}_i^{\intercal} \alpha_t \qquad \text{and} \qquad \mu^{0}_t(x_i) = \tilde{x}_i^{\intercal} \beta_t,
$$

and then

$$
\mu^{1/0}_t = \begin{bmatrix} \tilde{x}_1^{\intercal}\alpha \\ \vdots \\ \tilde{x}_{d_x}^{\intercal}\alpha \end{bmatrix} = \tilde{X}\alpha_t \qquad \text{and} \qquad \mu^{0}_t = \begin{bmatrix} \tilde{x}_1^{\intercal}\beta \\ \vdots \\ \tilde{x}_{d_x}^{\intercal}\beta \end{bmatrix} = \tilde{X}\beta_t.
$$

We then have following estimators of $\alpha^{\intercal}_{1:(T-1)} \equiv (\alpha^{\intercal}_1, \ldots, \alpha^{\intercal}_{T-1})$ and $\beta^{\intercal}_{2:(T-1)} \equiv (\beta^{\intercal}_2, \ldots, \beta^{\intercal}_{T-1})$:

$$
\hat{\alpha}_{1:(T-1)} = I_{T-1} \otimes (\tilde{X}^{\intercal}\tilde{X})^{-1} \tilde{X}^{\intercal} \hat{\mu}^{1/0}_{1:(T-1)}, \tag{5.6}
$$

$$
\hat{\beta}_{2:T} = I_{T-2} \otimes (\tilde{X}^{\intercal}\tilde{X})^{-1} \tilde{X}^{\intercal} \hat{\mu}^{0}_{2:(T-1)}, \tag{5.7}
$$

whose asymptotic variances are

$$\mathrm{Var}(\hat{\alpha}_{1:(T-1)}) = \left[I_{T-1} \otimes (\tilde{X}^\intercal \tilde{X})^{-1} \tilde{X}^\intercal\right] G^{1/0}_{1:T} \varPi (G^{1/0}_{1:T})^\intercal \left[I_{T-1} \otimes \tilde{X}(\tilde{X}^\intercal \tilde{X})^{-1}\right],$$
$$\mathrm{Var}(\hat{\beta}_{2:(T-1)}) = \left[I_{T-2} \otimes (\tilde{X}^\intercal \tilde{X})^{-1} \tilde{X}^\intercal\right] G^{0}_{1:T} \varPi (G^{0}_{1:T})^\intercal \left[I_{T-2} \otimes \tilde{X}(\tilde{X}^\intercal \tilde{X})^{-1}\right].$$

6 Numerical studies

In the numerical experiments below, we consider a stationary dynamic programming discrete choice model with a single X_t and a single excluded variable Z_t. In the Supplemental Material, we report the numerical studies for nonstationary models.

The support $\mathcal{X} \equiv \{ x_1, \dots x_{d_x} \}$ of X_t are the $d_x = 40$ cutting points that split the interval $[0, 2]$ into $d_x - 1$ equally spaced subintervals. The support of Z_t is $\mathcal{Z} \equiv \{ 1, \dots, d_z = 3 \}$. Let the state space $\mathcal{S} \equiv \mathcal{X} \times \mathcal{Z}$ and $d_s = d_x \cdot d_z$. The observable states $S_t = (X_t, Z_t)$ follows a homogenous controlled first-order Markov chain. For $\mathtt{d} \in \{ \mathtt{0}, \mathtt{1} \}$, let $F^{\mathtt{d}}$ be the time invariant $d_s \times d_s$ transition matrix describing the transition probability law from S_t to S_{t+1} given the discrete choice $D_t = \mathtt{d}$. The transition matrix $F^{\mathtt{d}}$ is randomly generated subjecting to the sparsity restriction that there are at most $m_s = 5$ number of states that can be reached in the next period. In the experiments below, let the discount factor $\delta = 0.8$. The per period utility functions are

$$\mu^{\mathtt{1}}(X) = 1 + X - X^2/2 \qquad \text{and} \qquad \mu^{\mathtt{0}}(X) = X.$$

The unobserved utilities shocks $\varepsilon^{\mathtt{0}}$ and $\varepsilon^{\mathtt{1}}$ are independent and follow the type 1 EVD. In the estimation below, we assume both the state transition matrices and the discount factor are known.

We compare our closed-form estimator with the three well known parametric estimators in the literature, including the nested fixed point (NFXP) algorithm (Rust, 1987), the pseudo-maximum likelihood (PML) estimator and the nested pseudo-likelihood (NPL) algorithm (Aguirregabiria and Mira, 2002). To implement their methods, we assume the parametric specification of the per period utility functions

$$\mu^{\mathtt{1/0}}(X_t) = \alpha_1 + \alpha_2 X_t + \alpha_3 X_t^2 \qquad \text{and} \qquad \mu^{\mathtt{0}}(X_t) = \beta_1 + \beta_2 X_t + \beta_3 X_t^2. \tag{6.1}$$

So the true values are $(\alpha_1, \alpha_2, \alpha_3) = (1, 0, -1/2)$ and $(\beta_1, \beta_2, \beta_3) = (0, 1, 0)$. We describe the implementation of these three estimators without using normalization assumption in the Supplemental Material.

Figure 6.1 shows the 95% confidence interval of the per period utility functions

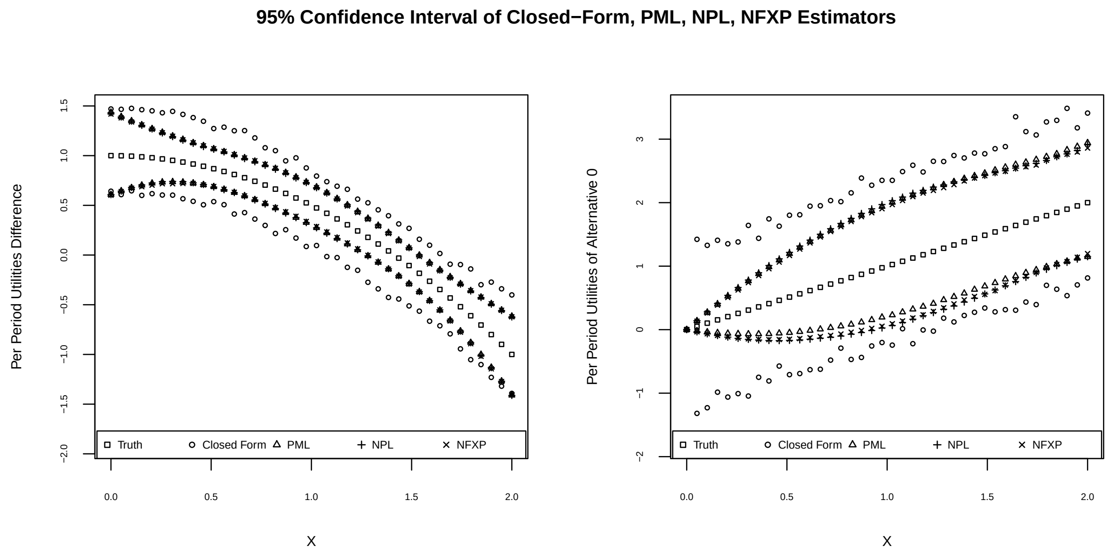

Figure 6.1: Estimation of Per Period Utilities in Stationary DPDC Models

based on our nonparametric closed form estimator, NFXP, PML and NPL. The estimates of the utility functions from NFXP, PML and NPL are based on the estimates of the parametric utility functions. Since our closed-form estimator does not have the information about the parametric form of the utility functions, its confidence interval is slightly wider than those of the parametric estimators, but the difference is marginal.

Using the parametric specification (6.1) and the nonparametric closed form estimates of $\mu^{1/0}$ and μ^0, we can estimate the unknown parameters α and β with the minimum distance estimator formula (5.3). Table 1 reports the estimation performance of the NFXP, PML, NPL and our minimum distance estimator. Our estimator ("Closed-Form") outperforms the other three estimators in terms of mean squared error (MSE) and computation time. Our estimator is 34, 600 and 850 times faster than the PML, NPL and NFXP algorithms, respectively. In addition to computation time, our estimator has smaller variance. In the experiments, we found that NFXP, PML and NPL are numerically unstable possibly due to the existence of multiple local maxima in the maximization of log likelihood function. Our estimator does not suffer from this issue because there is no numerical optimization involved at all.

We now consider the counterfactual intervention in the dynamic discrete choice model. In particular, we want to know how large would be the bias from using

Table 1: Estimation of Stationary DPDC Models

		$\alpha_1 = 1$	$\alpha_2 = 0$	$\alpha_3 = -0.5$	$\beta_1 = 0$	$\beta_2 = 1$	$\beta_3 = 0$	Time[1]
	Bias	0.008	0.003	-0.006	-	-0.017	0.014	
NFXP	Var.	0.044	0.218	0.051	-	0.704	0.152	*240*
	MSE	*0.044*	*0.218*	*0.051*	-	*0.704*	*0152*	
	Bias	0.008	-0.003	-0.003	-	0.158	-0.070	
PML	Var.	0.045	0.231	0.054	-	0.651	0.146	9.8
	MSE	*0.045*	*0.231*	*0.054*	-	*0.676*	*0.151*	
	Bias	0.004	0.003	-0.004	-	0.013	-0.003	
NPL	Var.	0.045	0.231	0.054	-	0.809	0.186	*170*
	MSE	*0.045*	*0.231*	*0.054*	-	*0.809*	*0.186*	
	Bias	0.103	-0.269	0.128	-0.048	0.172	-0.070	
Closed-Form[2]	Var.	0.032	0.122	0.029	0.147	0.299	0.059	*0.28*
	MSE	*0.043*	*0.194*	*0.045*	0.150	*0.328*	*0.064*	

Note: The results are based on 10 sets of state transition matrices and $1,000$ replications for each set. The cross-sectional sample size is $1,000$, and there is one period observation.

[1] The computation time is measured in second based on the average of the replications.

[2] We first estimate the per period utility functions nonparametrically using the formulas in equation (5.1) and (5.2). Then we estimate the parameters α and β by the formulas of equation (5.3).

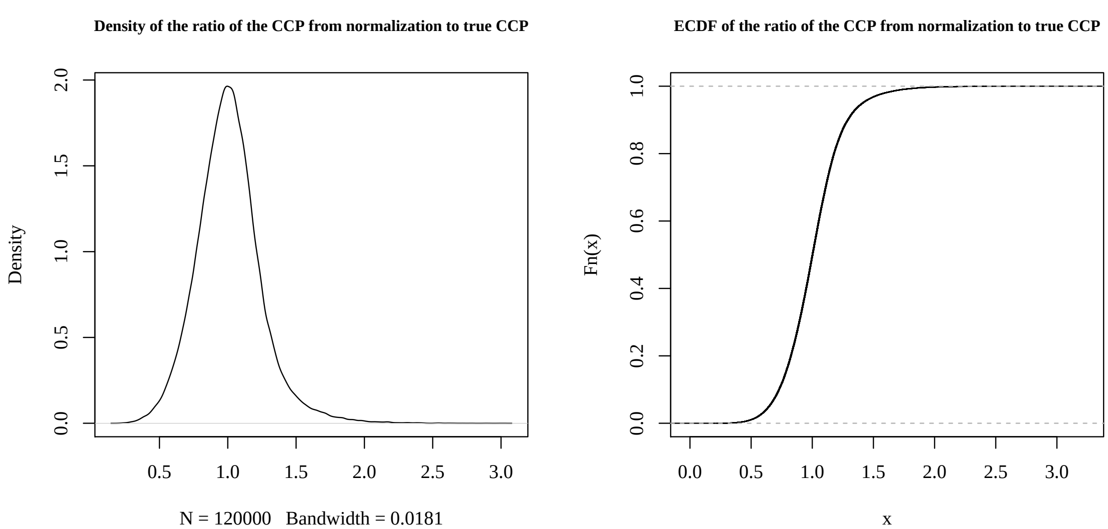

Figure 6.2: Density and Distribution of the Ratio of True Counterfactual CCP and Counterfactual CCP under Normalization

normalization $\mu^0(X) = 0$, which is wrong in this example since $\mu^0(x) = x$. Suppose $\tilde{F}^0$ and $\tilde{F}^1$ are counterfactual state transition matrices. Let $\tilde{p}$ be the true counterfactual CCP of the stationary DPDC model with everything unchanged excepting for that the state transition matrices are $\tilde{F}^0$ and $\tilde{F}^1$. Let $\tilde{p}_{\text{nm}}$ be the counterfactual CCP under the normalization assumption $\mu^0(X) = 0$. By trying $1,000$ pairs of state transition matrices (F^0, F^1) and $(\tilde{F}^0, \tilde{F}^1)$, we report the empirical density and CDF of the ratio $\tilde{p}(s_i)/\tilde{p}_{\text{nm}}(s_i)$ in figure 6.2. If the normalization $\mu^0(X) = 0$ incurs no bias, the ratio $\tilde{p}(s_i)/\tilde{p}_{\text{nm}}(s_i)$ always equals 1. However, this is not the case based on these two figures in figure 6.2. We can conclude based on figure 6.2 that the bias in counterfactual CCP from using normalization is noticeably large.

7 Female labor force participation example

In this empirical example, we studies how does husband's income growth affect wife's labor force participation decisions. In particular, we show how does the normalization of letting $\mu^0(S_t) = 0$ affect the women's counterfactual labor force participation probabilities. The data are the British Household Panel Survey (BHPS) from 1991-2009 (Wave 1-18). We use five periods data from wave 1, 5, 9, 13 and 17. So the first and the last sampling periods correspond to wave 1 and 17, respectively. We choose married women without college education.

From the BHPS data, we first obtain individuals' income and working experience that is measured by their accumulated number of working months in each period. Then for each period t and each married woman i, we construct three variables xp_{it} ($\text{xp}_{it} = k$, if woman i's working experience in period t is between the $(k-1)$-th and k-th quartile), xp_{it}^h ($\text{xp}_{it}^h = k$, if woman i's husband's working experience in period t is between the $(k-1)$-th and k-th quartile), and y_{it} ($y_{it} = k$, if woman i's husband's income in period t is between the $(k-1)$-th and k-th decile). We also observe woman i's labor force participation choice D_{it}. Let $s_{it} = (\text{xp}_{it}, \text{xp}_{it}^h, y_{it})$ be the vector of state variables, in which xp_{it}^h (the husband's working experience) is used as the excluded variable for identification. Note that s_{it} can take 160 different values.

We let woman i's per period utility depend on xp_{it}, y_{it} and D_{it}, but not on xp_{it}^h. We do not assign any parametric specification for the per period utility functions. In the estimation, we let the discount factor be a constant over time, and assume it is 0.8 corresponding to an annual discount rate of 0.95. The per period utility functions are estimated by the estimators in equation (5.4) and (5.5), which needs the estimation of the CCP and the state transition matrices first. The estimation of the CCP is based on the kernel estimators with Aitchison and Aitken (1976) discrete kernel functions. The estimation of the state transition matrices is more complicated. Without restriction on the conditional distribution $f(s_{i,t+1}|s_{it}, D_{it})$, there are 160^2 number of parameters for each period $t = 1, \dots, 4$. To circumvent the dimensionality problem, we assume that

$$f(s_{i,t+1}|s_{it}, D_{it}) = f(\mathrm{xp}_{i,t+1}|\mathrm{xp}_{it}, \mathrm{xp}^h_{it}, D_{it}) \cdot f(\mathrm{xp}^h_{i,t+1}|\mathrm{xp}^h_{it}, y_{it}, D_{it}) \cdot \\ f(y_{i,t+1}|\mathrm{xp}^h_{i,t+1}, \mathrm{xp}^h_{it}, y_{it}, D_{it}).$$

Still, each component, e.g. $f(y_{i,t+1}|\mathrm{xp}^h_{i,t+1}, \mathrm{xp}^h_{it}, y_{it}, D_{it})$, of the above decomposition involves a large number of parameters. We use the mixture transition distribution (MTD) models (Nicolau, 2014) in statistics to estimate each component. Take $f(y_{i,t+1}|\mathrm{xp}^h_{i,t+1}, \mathrm{xp}^h_{it}, y_{it}, D_{it})$ for example to explain the MTD model. For $j = 1, \dots, 10$, define

$$r(j, \mathrm{xp}^h_{i,t+1}, \mathrm{xp}^h_{it}, y_{it}, D_{it}) \equiv \\ (1, f(y_{i,t+1} = j|\mathrm{xp}^h_{i,t+1}, D_{it}), f(y_{i,t+1} = j|\mathrm{xp}^h_{it}, D_{it}), f(y_{i,t+1} = j|y_{it}, D_{it}))^\intercal.$$

The MTD model assumes that

$$f(y_{i,t+1} = j|\mathrm{xp}^h_{i,t+1}, \mathrm{xp}^h_{it}, y_{it}, D_{it}) = \frac{\Phi\left(r(j, \mathrm{xp}^h_{i,t+1}, \mathrm{xp}^h_{it}, y_{it}, D_{it})^\intercal \gamma_t\right)}{\sum_{k=1}^{10} \Phi\left(r(k, \mathrm{xp}^h_{i,t+1}, \mathrm{xp}^h_{it}, y_{it}, D_{it})^\intercal \gamma_t\right)},$$

where $\Phi(\cdot)$ is the CDF of the standard normal distribution. The unknown parameter γ_t is estimated by the MLE after estimating the conditional probabilities, e.g. $f(y_{i,t+1}|y_{it}, D_{it})$, in $r(j, \mathrm{xp}^h_{i,t+1}, \mathrm{xp}^h_{it}, y_{it}, D_{it})$.

We consider the counterfactual change in husbands' income growth. The conditional probability function $f(y_{i,t+1}|\mathrm{xp}^h_{i,t+1}, \mathrm{xp}^h_{it}, y_{it}, D_{it})$ in the MTD model is determined by the transition probabilities in the vector $r(j, \mathrm{xp}^h_{i,t+1}, \mathrm{xp}^h_{it}, y_{it}, D_{it})$ and the coefficients γ_t. Let $\hat{\gamma}_t$ be the estimate of γ_t, and let $\hat{f}(y_{i,t+1}|\mathrm{xp}^h_{i,t+1}, D_{it})$, $\hat{f}(y_{i,t+1}|\mathrm{xp}^h_{it}, D_{it})$ and $\hat{f}(y_{i,t+1}|y_{it}, D_{it})$ be the estimates of the conditional probability functions. Our counterfactual experiment is to define the counterfactual transition probability function

$$\tilde{f}(y_{i,t+1}|y_{it}, D_{it}) = \begin{cases} \hat{f}(y_{i,t+1}|y_{it}, D_{it}), & \text{if } y_{i,t+1} < y_{it}, \\ \sum_{j=y_{it}}^{10} \hat{f}(y_{i,t+1} = j|y_{it}, D_{it}), & \text{if } y_{i,t+1} = y_{it}, \\ 0, & \text{if } y_{i,t+1} > y_{it}, \end{cases}$$

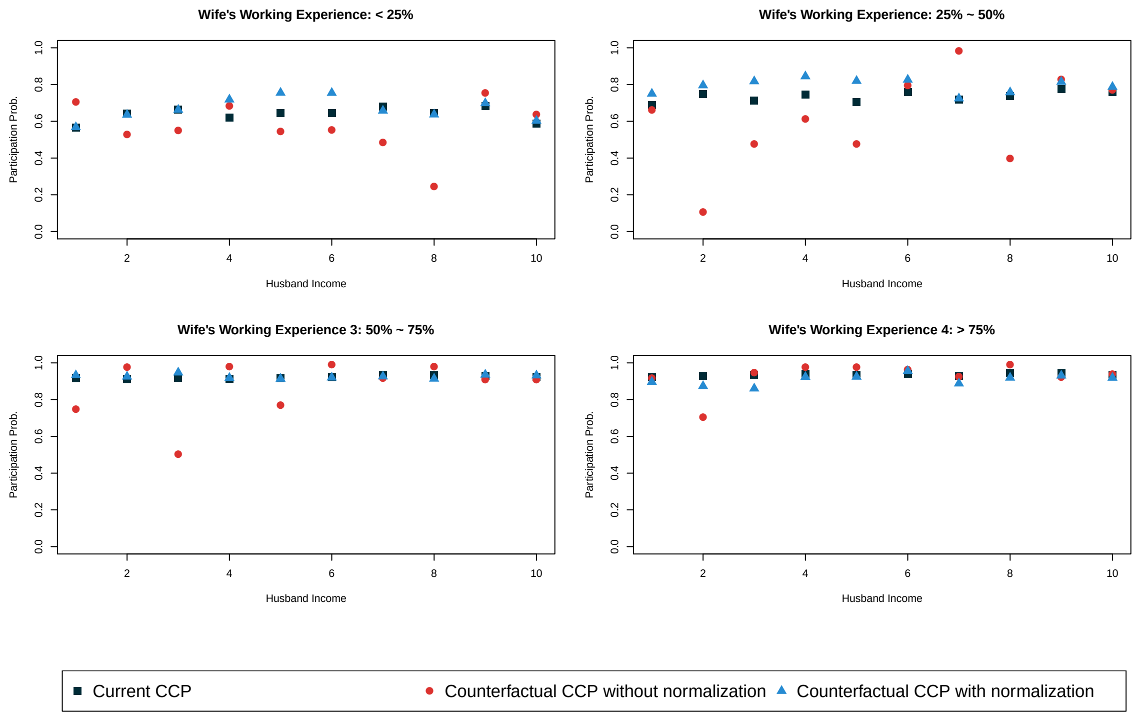

Figure 7.1: Counterfactual female labor force participation probabilities

for $t = 1, 2, 3$, and let $\tilde{f}(y_{i,t+1}|y_{it}, D_{it}) = \hat{f}(y_{i,t+1}|y_{it}, D_{it})$ for $t = 4$. This implies that husband's income decile cannot move upward. Then we form the counterfactual probability $\tilde{f}(y_{i,t+1}|\text{xp}^h_{i,t+1}, \text{xp}^h_{it}, y_{it}, D_{it})$ by let

$$\tilde{r}(j, \text{xp}^h_{i,t+1}, \text{xp}^h_{it}, y_{it}, D_{it}) \equiv \\ (1, \hat{f}(y_{i,t+1} = j|\text{xp}^h_{i,t+1}, D_{it}), \hat{f}(y_{i,t+1} = j|\text{xp}^h_{it}, D_{it}), \tilde{f}(y_{i,t+1} = j|y_{it}, D_{it}))^\intercal,$$

$$\tilde{f}(y_{i,t+1} = j|\text{xp}^h_{i,t+1}, \text{xp}^h_{it}, y_{it}, D_{it}) = \frac{\Phi\left(\tilde{r}(j, \text{xp}^h_{i,t+1}, \text{xp}^h_{it}, y_{it}, D_{it})^\intercal \hat{\gamma}_t\right)}{\sum_{k=1}^{10} \Phi\left(\tilde{r}(k, \text{xp}^h_{i,t+1}, \text{xp}^h_{it}, y_{it}, D_{it})^\intercal \hat{\gamma}_t\right)}.$$

We do not change $f(\text{xp}_{i,t+1}|\text{xp}_{it}, \text{xp}^h_{it}, D_{it})$ or $f(\text{xp}^h_{i,t+1}|\text{xp}^h_{it}, y_{it}, D_{it})$ in this experiment. So the counterfactual state transition law is

$$\tilde{f}(s_{i,t+1}|s_{it}, D_{it}) = \hat{f}(\text{xp}_{i,t+1}|\text{xp}_{it}, \text{xp}^h_{it}, D_{it}) \cdot \hat{f}(\text{xp}^h_{i,t+1}|\text{xp}^h_{it}, y_{it}, D_{it}) \cdot \\ \tilde{f}(y_{i,t+1}|\text{xp}^h_{i,t+1}, \text{xp}^h_{it}, y_{it}, D_{it}).$$

We want to know the counterfactual CCP in period 1, 2 and 3 given the above counterfactual state transition law. We also estimate the counterfactual CCP under the normalization assumption such that $\mu_t^0 = 0$ for $t = 1, \ldots, 5$.

Figure 7.1 shows the counterfactual female labor force participation probabilities in wave 1 with $\mathrm{xp}_{it}^h = 3$ (i.e. husband's working experience is between the second and the third quartile). The results for the other waves and husband's working experience are similar and reported in the Supplemental Material. It is interesting to observe that without normalization, the counterfactual labor force participation probabilities are lower than the actual ones for the women whose working experience is below the median. With normalization, however, the counterfactual labor force participation probabilities seems to be close to the actual ones, suggesting that the husbands' income stagnation has no effects on female labor force participation. It is interesting to note that in real economy, the labor force participation rate decreases as the earnings grow slowly. The percent changes in average hourly real earnings in 2008 (January) and 2016 (June) are 3.7% and 1.7% in the United States, according to the Current Employment Statistics; the labor force participation rates in 2008 (January) and 2016 (January) are 66.2% and 62.7% in the United States, according to the Current Population Survey.

8 Concluding remarks

The identification and estimation of DPDC models are considered to be complicated and numerically difficult. This paper shows that the identification of DPDC model is indeed equivalent to the identification of a linear GMM system. So the identification and estimation of DPDC models become easy to address. We show how to identify DPDC models under a variety of restrictions. In particular, we show how to identify the DPDC model without normalizing the per period utility function of any alternative. This case is particularly important because the normalization of per period utility functions can usually bias the counterfactual policy predictions. Due to the equivalence to a linear GMM system, we can propose a closed form nonparametric estimator for the per period utility functions without using any terminal conditions or assuming the dynamic programming problem is stationary. The implementation of our estimator does not involve any numerical optimization. So it is numerically stabler and faster than the existing estimators, such as NFXP, PML and NPL.

APPENDIX

A Proofs

Lemma A.1. *Let A be an $m \times n$ real matrix with $m \geq n-1$. Suppose each row of A sums to be zero and* $\operatorname{rank} A = n-1$. *Suppose the linear equation $Ax = b$ has solutions. Then the solution set is $\{A^+b + c \times 1_n : \forall c \in \mathbb{R}\}$, where A^+ is the Moore-Penrose pseudoinverse of A, and 1_n is a n-dimensional vector of ones.*

Proof. We know that the solution set of equation $Ax = b$ is $\{A^+b + (I_n - A^+A)a : \forall a \in \mathbb{R}^n\}$. It suffices to show that $(I_n - A^+A)$ is an $n \times n$ matrix, whose elements are identical.

Let $A = U\Sigma V^\intercal$ be an singular value decomposition (SVD) of matrix A. We know that $A^+ = V\Sigma^+U^\intercal$, where Σ^+ is the pseudoinverse of Σ. Because U and V are both orthogonal matrices, we have $A^+A = V\Sigma^+\Sigma V^\intercal$ as an eigenvalue decomposition (EVD). When $\operatorname{rank} A = n-1$, we have that $\Sigma^+\Sigma$ is a $n \times n$ diagonal matrix, of which the vector of the main diagonal entries is $(1_{n-1}^\intercal, 0)^\intercal$. So the columns of V are eigenvectors of A^+A corresponding to the eigenvalues 1 and 0. Because the sum of each row of A is zero, 1_n is an eigenvector of A^+A corresponding to eigenvalue zero, and $n^{-1/2} \cdot 1_n$ is one column of V. Removing the column $n^{-1/2} \cdot 1_n$ from matrix V, we obtain an $n \times (n-1)$ matrix $\tilde{V}$ and $A^+A = V\Sigma^+\Sigma V^\intercal = \tilde{V}\tilde{V}^\intercal$.

Because V is an orthogonal matrix, we have

$$\begin{aligned}
I = VV^\intercal &= \begin{bmatrix} \tilde{V} & n^{-1/2} \times 1_n \end{bmatrix} \begin{bmatrix} \tilde{V}^\intercal \\ n^{-1/2} \times 1_n^\intercal \end{bmatrix} \\
&= \tilde{V}\tilde{V}^\intercal + n^{-1} \cdot 1_n {1_n}^\intercal \\
&= A^+A + n^{-1} \cdot 1_{n\times n}.
\end{aligned}$$

Here $1_{n\times n}$ is a $n \times n$ matrix whose elements are all 1. So we have $I - A^+A = n^{-1} \cdot 1_{n\times n}$, and the lemma follows. □

Lemma A.2. *Let A_1 and A_2 both be $m \times n$ real matrices with $m \geq 2n-2$. Define a block matrix $A \equiv \left[\begin{array}{c|c} A_1 & A_2 \end{array}\right]$. For each $i = 1, 2$, suppose each row of A_i sums to be zero, and* $\operatorname{rank} A = 2n-2$. *Suppose linear equation $Ax = b$ has solutions. Then the solution set of the equation is $\{\, A^+b + (c_1 \cdot 1_n^\intercal, c_2 \cdot 1_n^\intercal)^\intercal : c_1, c_2 \in \mathbb{R} \,\}$.*

Proof. The proof is similar to the proof of lemma A.1. The solution set of equation $Ax = b$ is $\{A^+b + (I_{2n} - A^+A)a : \forall a \in \mathbb{R}^{2n}\}$. Let $A = U\Sigma V^\intercal$ be an SVD of matrix A. We have $A^+A = V\Sigma^+\Sigma V^\intercal$ as an EVD of A^+A. Because $\operatorname{rank} A = 2n-2$

and the row sums of each A_i $(i = 1, 2)$ are zero, we have that $\Sigma^+\Sigma$ is a $2n \times 2n$ diagonal matrix, of which the vector of the main diagonal entries is $(1_{2n-2}^\intercal, 0, 0)^\intercal$. So V has two columns $w_1^\intercal = n^{-1/2} \cdot (1_n^\intercal, 0_n^\intercal)$ and $w_2^\intercal = n^{-1/2} \cdot (0_n^\intercal, 1_n^\intercal)$, because they are two orthonormal eigenvectors corresponding to eigenvalue 0. Removing w_1 and w_2 from the columns of matrix V, we obtain an $2n \times (2n-2)$ matrix $\tilde{V}$ whose columns are eigenvectors corresponding to the $2n-2$ nonzero eigenvalues. We then have $A^+A = V\Sigma^+\Sigma V^\intercal = \tilde{V}\tilde{V}^\intercal$.

Because V is an orthogonal matrix, we have

$$
\begin{aligned}
I = VV^\intercal = \begin{bmatrix} \tilde{V} & w_1 & w_2 \end{bmatrix} \begin{bmatrix} \tilde{V}^\intercal \\ w_1^\intercal \\ w_2^\intercal \end{bmatrix} \\
= \tilde{V}\tilde{V}^\intercal + w_1 w_1{}^\intercal + w_2 w_2^\intercal \\
= A^+A + \begin{bmatrix} 1_{n\times n} & \\ & 1_{n\times n} \end{bmatrix}.
\end{aligned}
$$

The rest of the proof follows immediately. $\square$

Proof of Proposition 1. Equation (4.9) is equivalent to the following,

$$
\begin{aligned}
\mu_1^{1/0} \otimes 1_{d_z} + F_2^{1/0}(\delta_1 \cdot v_2) = \phi(p_1), \\
\left(\prod_{r=1}^{t-1} \delta_r\right) \cdot \mu_t^{1/0} \otimes 1_{d_z} + \left(\prod_{r=1}^{t} \delta_r\right) \cdot F_{t+1}^{1/0} v_{t+1} = \left(\prod_{r=1}^{t-1} \delta_r\right) \cdot \phi(p_t), \\
\left(\prod_{r=1}^{t-1} \delta_r\right) \cdot v_t - \left(\prod_{r=1}^{t-1} \delta_r\right) \cdot \mu_t^0 \otimes 1_{d_z} - \left(\prod_{r=1}^{t} \delta_r\right) \cdot F_{t+1}^0 v_{t+1} = \left(\prod_{r=1}^{t-1} \delta_r\right) \cdot \psi(p_t),
\end{aligned}
\tag{A.1}
$$

for $t = 2, \ldots, T-1$. In the remainder of the proof, we derive the explicit solutions of $\mu_t^{1/0}$ and μ_t^0.

Multiplying both sides of the equations in (A.1) with the M matrix defined by (4.14), we have

$$
A_{1:T} \begin{bmatrix} \delta_1 \cdot v_2 \\ (\prod_{r=1}^2 \delta_r) \cdot v_3 \\ \vdots \\ (\prod_{r=1}^{T-1} \delta_r) \cdot v_T \end{bmatrix} = b_{1:T}, \tag{A.2}
$$

where $A_{1:T}$ and $b_{1:T}$ are as defined in the proposition. Note that $A_{1:T}^+ b_{1:T}$ is one solution of equation (A.2).

It follows from equation (A.1) that

$$\begin{bmatrix} \mu_1^{1/0} \otimes 1_{d_z} \\ (\prod_{r=1}^{1} \delta_r) \cdot \mu_2^{1/0} \otimes 1_{d_z} \\ \vdots \\ (\prod_{r=1}^{T-2} \delta_r) \cdot \mu_{T-1}^{1/0} \otimes 1_{d_z} \end{bmatrix} = (\tilde{\Lambda} \otimes I_{d_s}) \begin{bmatrix} \phi(p_1) \\ \phi(p_2) \\ \vdots \\ \phi(p_{T-1}) \end{bmatrix} - F_{2:T}^{1/0} \begin{bmatrix} \delta_1 \cdot v_2 \\ (\prod_{r=1}^{2} \delta_r) \cdot v_3 \\ \vdots \\ (\prod_{r=1}^{T-1} \delta_r) \cdot v_T \end{bmatrix}, \quad \text{(A.3)}$$

$$\begin{bmatrix} \delta_1 \cdot \mu_2^{0} \otimes 1_{d_z} \\ \vdots \\ (\prod_{r=1}^{T-2} \delta_r) \cdot \mu_{T-1}^{0} \otimes 1_{d_z} \end{bmatrix} = \check{F}_{3:T}^{0} \begin{bmatrix} \delta_1 \cdot v_2 \\ \vdots \\ (\prod_{r=1}^{T-1} \delta_r) \cdot v_T \end{bmatrix} - (\Lambda \otimes I_{d_s}) \begin{bmatrix} \psi(p_2) \\ \vdots \\ \psi(p_{T-1}) \end{bmatrix}. \quad \text{(A.4)}$$

Substituting $(\delta_1 \cdot v_2, \ldots, (\prod_{r=1}^{T-1} \delta_r) \cdot v_T)^\intercal$ in equation (A.3) with $A_{1:T}^{+} b_{1:T}$, and multiplying both sides of equation (A.3) with $\tilde{\Lambda}^{-1} \otimes I_{d_s}$, we have

$$\begin{bmatrix} \mu_1^{1/0} \otimes 1_{d_z} \\ \vdots \\ \mu_{T-1}^{1/0} \otimes 1_{d_z} \end{bmatrix} = \begin{bmatrix} \phi(p_1) \\ \vdots \\ \phi(p_{T-1}) \end{bmatrix} - (\tilde{\Lambda}^{-1} \otimes I_{d_s}) F_{2:T}^{1/0} A_{1:T}^{+} b_{1:T}. \quad \text{(A.5)}$$

Similar operations for equation (A.4) gives

$$\begin{bmatrix} \mu_2^{0} \otimes 1_{d_z} \\ \vdots \\ \mu_{T-1}^{0} \otimes 1_{d_z} \end{bmatrix} = (\Lambda^{-1} \otimes I_{d_s}) \check{F}_{3:T}^{0} A_{1:T}^{+} b_{1:T} - \begin{bmatrix} \psi(p_2) \\ \vdots \\ \psi(p_{T-1}) \end{bmatrix}. \quad \text{(A.6)}$$

Multiplying both sides of equation (A.5) (equation (A.6)) with $I_{T-1} \otimes W$ $(I_{T-2} \otimes (WL))$, we have formula (4.15) (formula (3.17)) in the proposition. Here we used $(I_{T-2} \otimes W)(I_{T-2} \otimes L) = I_{T-2} \otimes (WL)$. □

References

Aguirregabiria, Victor. 2010. "Another look at the identification of dynamic discrete decision processes: An application to retirement behavior." *Journal of Business & Economic Statistics* 28 (2):201–218.

Aguirregabiria, Victor and Pedro Mira. 2002. "Swapping the nested fixed point algorithm: A class of estimators for discrete Markov decision models." *Econometrica* 70 (4):1519–1543.

Aguirregabiria, Victor and Junichi Suzuki. 2014. "Identification and counterfactuals in dynamic models of market entry and exit." *Quantitative Marketing and Economics* 12 (3):267–304.

Aitchison, John and Colin GG Aitken. 1976. "Multivariate binary discrimination by the kernel method." *Biometrika* 63 (3):413–420.

Arcidiacono, Peter and Robert A Miller. 2015. "Identifying Dynamic Discrete Choice Models off Short Panels." Working paper.

Bajari, Patrick, C Lanier Benkard, and Jonathan Levin. 2007. "Estimating dynamic models of imperfect competition." *Econometrica* 75 (5):1331–1370.

Bajari, Patrick, Victor Chernozhukov, Han Hong, and Denis Nekipelov. 2009. "Identification and efficient semiparametric estimation of a dynamic discrete game." Tech. rep., National Bureau of Economic Research.

Bajari, Patrick, Han Hong, and Denis Nekipelov. 2010. "Game theory and econometrics: A survey of some recent research." In *Advances in Economics and Econometrics: Tenth World Congress*, vol. 3. 3–52.

Blevins, Jason R. 2014. "Nonparametric identification of dynamic decision processes with discrete and continuous choices." *Quantitative Economics* 5 (3):531–554.

Blundell, Richard W and James L Powell. 2004. "Endogeneity in Semiparametric Binary Response Models." *The Review of Economic Studies* 71 (3):655–679.

Ching, Andrew and Matthew Osborne. 2015. "Identification and Estimation of Forward-Looking Behavior: The Case of Consumer Stockpiling." *Available at SSRN 2594032* .

Fang, Hanming and Yang Wang. 2015. "Estimating dynamic discrete choice models with hyperbolic discounting, with an application to mammography decisions." *International Economic Review* 56 (2):565–596.

Hotz, V Joseph and Robert A Miller. 1993. "Conditional choice probabilities and the estimation of dynamic models." *The Review of Economic Studies* 60 (3):497–529.

Hu, Yingyao and Matthew Shum. 2012. "Nonparametric identification of dynamic models with unobserved state variables." *Journal of Econometrics* 171 (1):32–44.

Imbens, Guido W and Whitney K Newey. 2009. "Identification and Estimation of Triangular Simultaneous Equations Models without Additivity." *Econometrica* 77 (5):1481–1512.

Kalouptsidi, Myrto, Paul T Scott, and Eduardo Souza-Rodrigues. 2015. "Identification of Counterfactuals and Payoffs in Dynamic Discrete Choice with an Application to Land Use." Working paper, NBER.

Kasahara, Hiroyuki and Katsumi Shimotsu. 2009. "Nonparametric Identification of Finite Mixture Models of Dynamic Discrete Choices." *Econometrica* 77 (1):135–175.

Keane, Michael P, Petra E Todd, and Kenneth I Wolpin. 2011. "The structural estimation of behavioral models: Discrete choice dynamic programming methods and applications." In *Handbook of Labor Economics*, vol. 4a, edited by Orley Ashenfelter and David Card, chap. 4. Elsevier, 331–461.

Magnac, Thierry and David Thesmar. 2002. "Identifying dynamic discrete decision processes." *Econometrica* 70 (2):801–816.

Matzkin, Rosa L. 1992. "Nonparametric and distribution-free estimation of the binary threshold crossing and the binary choice models." *Econometrica* :239–270.

Nicolau, João. 2014. "A New Model for Multivariate Markov Chains." *Scandinavian Journal of Statistics* 41 (4):1124–1135.

Norets, Andriy and Xun Tang. 2014. "Semiparametric Inference in dynamic binary choice models." *The Review of Economic Studies* 81:1229–1262.

Pesendorfer, Martin and Philipp Schmidt-Dengler. 2008. "Asymptotic least squares estimators for dynamic games." *The Review of Economic Studies* 75 (3):901–928.

Rust, John. 1987. "Optimal replacement of GMC bus engines: An empirical model of Harold Zurcher." *Econometrica* :999–1033.

———. 1994. "Structural estimation of Markov decision processes." In *Handbook of econometrics*, vol. 4, edited by R. F. Engle and D.L. McFadden, chap. 51. Elsevier, 3081–3143.

Srisuma, Sorawoot and Oliver Linton. 2012. "Semiparametric estimation of Markov decision processes with continuous state space." *Journal of Econometrics* 166 (2):320–341.

Su, Che-Lin and Kenneth L Judd. 2012. "Constrained optimization approaches to estimation of structural models." *Econometrica* 80 (5):2213–2230.